On statistical convergence of order $\alpha$ of difference sequence of functions

Muhammed Cinar $^1,$* and Mikail Et $^2$

$^1$ Department of Mathematics, Mus Alparslan University, Mus-Turkey
$^2$ Department of Mathematics, Firat University, 23119, Elazig-Turkey

* Corresponding author, e-mail: muhammedcinar23@gmail.com

Received: 19 December 2014 / Accepted: 13 April 2016 / Published: 21 April 2016

Abstract: In this paper we introduce the concepts of uniform and pointwise $\Delta_m^\alpha$ statistical convergence of order $\alpha$ of sequence of functions. We also give notions of uniform and pointwise $\Delta_m^\alpha$ statistically Cauchy sequence of order $\alpha$ of sequence of functions and show that pointwise $\Delta_m^\alpha$ statistically Cauchy sequence (uniform $\Delta_m^\alpha$ statistically convergence) is equivalent to pointwise $\Delta_m^\alpha$ statistically convergence (uniform $\Delta_m^\alpha$ statistically Cauchy sequence) of order $\alpha$. Moreover, some relations between pointwise $\Delta_m^\alpha$ statistically convergence of order $\alpha$ and strongly pointwise $\Delta_q^\alpha$ Cesàro summable of order $\alpha$ of sequence of functions are given.

Key words: statistical convergence, sequence of functions, Cesàro summability

INTRODUCTION

Fast [1] and Schoenberg [2] independently introduced the notion of statistical convergence. The idea hinges on the density of subsets of the set $\mathbb{N}$. The density of $E$, a subset of $\mathbb{N}$, is defined by $\delta(E) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{E}(k)$, provided that the limit exists, where $\chi_{E}$ is the characteristic function of $E$. A sequence $x = (x_k)$ is statistically convergent to a number $L$ if, for every $\varepsilon > 0$, $\delta\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} = 0$. The statistical convergence has been studied by Basar [3], Connor [4], Cinar et al. [5], Et et al. [6-7], Duman and Ankara [8], Fridy [9], Gokhan and Gungor [10], Gungor et al. [11], Gungor and Gokhan [12, 13], Mohiuddine et al. [14], Mursaleen [15], Salat [16] and many others.

Recently Gadjiev and Orhan [17] introduced the order of statistical convergence of a sequence of numbers and later Colak [18] studied statistical convergence of order $\alpha$ and strong $q$–Cesàro summability of order $\alpha$ and defined the $\alpha$–density of a subset $E$ of $\mathbb{N}$ as
\[ \delta_\alpha (E) = \lim_{n} \frac{1}{n^{\alpha}} \left| \{ k \leq n : k \in E \} \right|, \]
provided that the limit exists and \( \alpha \in (0,1) \),
where \( \left| \{ k \leq n : k \in E \} \right| \) denotes the number of elements of \( E \) not exceeding \( n \).

If \( x = (x_k) \) is a sequence such that \( x_k \) satisfies the property \( P(k) \) for all \( k \) except a set of zero \( \alpha - \text{density} \), then we say that \( x_k \) satisfies the property \( P(k) \) for ‘almost all \( k \) according to \( \alpha \)’ and we abbreviate this by ‘\( a.a.k (\alpha) \)’. It can be shown that any finite subset of \( \mathbb{N} \) has zero \( \alpha \) density and \( \delta_\alpha (E^c) = 1 - \delta_\alpha (E) \) does not hold for \( \alpha \in (0,1) \). If \( \alpha = 1 \), then \( \alpha - \text{density} \) reduces to the natural density.

Kizmaz [19] introduced the difference sequence spaces and Et and Colak [20] generalised the notion afterwards as follows:

\[ \Delta^n (X) = \{ x = (x_k) : (\Delta^n x_k)_k \in X \}, \]
for \( X = \ell_\alpha, c \) or \( c_0 \), where \( m \in \mathbb{N}, \Delta^0 x = (x_k), \Delta^n x = (\Delta^{n-1} x_k - \Delta^{n-1} x_{k+1}) \) and so \( \Delta^n x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i} \).
Recently, Basar and Altay [21], Altay and Basar [22], Altin et al. [23], Et [24], Altinok and Mursaleen [25], Bektas et al. [26], Colak et al. [27], Djolovic and Malkowsky [28], Gunorg and Et [29], Malkowsky et al. [30] and Mursaleen et al. [31] have studied difference sequences spaces.

**MAIN RESULTS**

In this section we give the main results of this article.

**Definition 1.** Let \( A \) be a subset of \( \mathbb{R} \) and \( \{ f_k \} \) be a sequence of real valued functions defined on \( A \). The pointwise \( \Delta^n \) – convergent sequence \( \{ f_k \} \) converges to \( f \) if there exists a natural number \( N = N(x, \varepsilon) \) for \( x \in A \) and \( \varepsilon > 0 \) such that \( \left| \Delta^n f_k (x) - f(x) \right| < \varepsilon \) for every \( k \geq N \). The notation \( N = N(x, \varepsilon) \) means that the natural number \( N \) depends on \( x \) and \( \varepsilon \). In this case we write \( \lim_k \Delta^n f_k (x) = f(x) \) for every \( x \in A \). The set of all pointwise \( \Delta^n \) – convergent sequences of functions is denoted by \( c(\Delta^n, F(p)) \), where \( \Delta^n f_k (x) = \sum_{v=0}^{m} (-1)^v \binom{m}{v} f_{k+v}(x) \).

**Definition 2.** Let \( A \) be a subset of \( \mathbb{R}, \{ f_k \} \) be a sequence of real valued functions defined on \( A \), and \( \alpha \) be any real number such that \( 0 < \alpha \leq 1 \). The sequence \( \{ f_k \} \) is said to be pointwise \( \Delta^n \) – statistically convergent of order \( \alpha \) to \( f \) if there exists a natural number \( N = N(x, \varepsilon) \) for \( x \in A \) and \( \varepsilon > 0 \) such that \( \left| \Delta^n f_k (x) - f(x) \right| < \varepsilon \ a.a.k(\alpha) \); that is,

\[ \lim_{n} \frac{1}{n^{\alpha}} \left| \{ k \leq n : \left| \Delta^n f_k (x) - f(x) \right| \geq \varepsilon \} \right| = 0. \]

In this case we write \( S^\alpha \lim \Delta^n f_k (x) = f(x) \) on \( A \). The set of all pointwise \( \Delta^n \) – statistically convergent sequences of functions of order \( \alpha \) is denoted by \( S^\alpha (\Delta^n, F(p)) \).

It can be shown that the pointwise \( \Delta^n \) – statistical convergence of order \( \alpha \) for sequence of
Therefore, then indeed if we take as desired. In this case, we have statistically convergent of order \( \alpha \), \( \lim_{n \to \infty} n^{-\alpha} \) for \( \alpha > 1 \). For this, let \( \{f_k\} \) be defined as follows:

\[
f_k(x) = \begin{cases} 
  1 & k = 2n \\
  x^{k-n} & k \neq 2n 
\end{cases} \quad n = 1, 2, 3, \ldots, x \in [0, \frac{1}{2}].
\]

Then we calculate \( \Delta f_k(x) \) as

\[
\Delta f_k(x) = \begin{cases} 
  1 - x^{k+1} & k = 2n \\
  x^{k-1} & k \neq 2n 
\end{cases} \quad n = 1, 2, 3, \ldots, x \in [0, \frac{1}{2}].
\]

In this case

\[
\lim_{n \to \infty} n^{-\alpha} \left| k \leq n : \left| \Delta f_k(x) - (x^{k-1}) \right| \geq \varepsilon, \text{ for every } x \in A \right| = \lim_{n \to \infty} \frac{n}{2n^\alpha} = 0
\]

and

\[
\lim_{n \to \infty} n^{-\alpha} \left| k \leq n : \left| \Delta f_k(x) - (1 - x^{k+1}) \right| \geq \varepsilon, \text{ for every } x \in A \right| = \lim_{n \to \infty} \frac{n}{2n^\alpha} = 0
\]

for \( \alpha > 1 \). Therefore, \( S^\alpha - \lim \Delta f_k(x) = 1 \) and \( S^\alpha - \lim \Delta f_k(x) = -1 \), which is impossible.

**Theorem 1.** Let \( \alpha \in (0, 1] \), and \( \{f_k\} \) and \( \{g_k\} \) be two sequences of real valued functions defined on \( A \).

(i) If \( S^\alpha - \lim \Delta^m f_k(x) = f(x) \) and \( c \in \mathbb{R} \), then \( S^\alpha - \lim \Delta^m f_k(x) = cf(x) \).

(ii) If \( S^\alpha - \lim \Delta^m f_k(x) = f(x) \) and \( S^\alpha - \lim \Delta^m g_k(x) = g(x) \), then

\[
S^\alpha - \lim \Delta^m (f_k(x) + g_k(x)) = f(x) + g(x)
\]

**Proof.** Let \( \{f_k\} \) and \( \{g_k\} \in S^\alpha (\Delta^m, F(p)) \) with \( S^\alpha - \lim \Delta^m f_k(x) = f(x) \) and \( S^\alpha - \lim \Delta^m g_k(x) = g(x) \) on \( A \subset \mathbb{R} \), and \( c \in \mathbb{R} \). Then one can easily see by these assumptions that

\[
S^\alpha - \lim \Delta^m (f_k(x) + g_k(x)) = f(x) + g(x),
\]

as desired.

It can be shown that every pointwise \( \Delta^m \) – convergent sequence of functions is pointwise \( \Delta^m \) – statistically convergent of order \( \alpha \) \((0 < \alpha \leq 1)\). However, the converse of this does not hold. Indeed if we take the sequence \( \{f_k\} \) defined by

\[
f_k(x) = \begin{cases} 
  1 & k = n^2 \\
  \frac{kx}{1 + k^2 x^2} & \text{otherwise}
\end{cases}
\]

then we have

\[
\Delta f_k(x) = \begin{cases} 
  1 - \frac{(k+1)x}{1 + (k+1)^2 x^2} & k = n^2 \\
  \frac{kx}{1 + k^2 x^2} - 1 & k = n^2 - 1 \\
  \frac{kx}{1 + k^2 x^2} - \frac{(k+1)x}{1 + (k+1)^2 x^2} & \text{otherwise}
\end{cases}
\]

Therefore, \( \{f_k\} \) is pointwise \( \Delta \) – statistically convergent of order \( \alpha \) with \( S^\alpha - \lim \Delta f_k(x) = 0 \) for \( \alpha > \frac{1}{2} \), but it is not pointwise \( \Delta \) – convergent.

**Definition 3.** Let \( A \) be a subset of \( \mathbb{R} \), \( \{f_k\} \) be a sequence of real valued functions defined on \( A \),
and $\alpha$ be any real number such that $0 < \alpha \leq 1$. The sequence $\{f_k\}$ is a pointwise $\Delta^n$–statistically Cauchy sequence of order $\alpha$ provided that there exists a number $N = N(\varepsilon, x)$ for $x \in A$ and $\varepsilon > 0$ such that

$$|\Delta^m f_k(x) - \Delta^m f_N(x)| < \varepsilon \quad a.a.k \ (\alpha);$$

that is, for $x \in A$ and $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{n^{\alpha}} \sum_{k \leq n} |\Delta^m f_k(x) - \Delta^m f_N(x)| \geq \varepsilon = 0.$$

Using the techniques for proving Theorem 3.4 in Cinar et al. [5], we obtain the proof of the following theorem.

**Theorem 2.** Let $\{f_k\}$ be a sequence of functions defined on a set $A$. The following statements are equivalent:

(i) $\{f_k\}$ is a pointwise $\Delta^n$–statistically convergent sequence of order $\alpha$ on $A$;

(ii) $\{f_k\}$ is a pointwise $\Delta^n$–statistically Cauchy sequence of order $\alpha$ on $A$;

(iii) $\{f_k\}$ is a sequence of functions for which there is a pointwise convergent sequence of function $\{\Delta^n g_k\}$ such that $\Delta^n f_k(x) = \Delta^n g_k(x)$ a.a.$k \ (\alpha)$ for every $x \in A$.

**Theorem 3.** If $0 < \alpha \leq \beta \leq 1$, then the inclusion $S^\alpha(\Delta^n, F(p)) \subseteq S^\beta(\Delta^n, F(p))$ strictly holds.

**Proof.** Since the proof of the inclusion $S^\alpha(\Delta^n, F(p)) \subseteq S^\beta(\Delta^n, F(p))$ is easy, we omit detail. To show the strictness of the inclusion, consider a sequence $\{f_k\}$ defined by

$$f_k(x) = \begin{cases} 1, & k = n^2, \\
\frac{k^2}{1+k^2}, & \text{otherwise} \\
n = 1, 2, 3, ..., x \in [0,1].
\end{cases}$$

Then one can calculate $\Delta f_k(x)$ as follows:

$$\Delta f_k(x) = \begin{cases} 1 - \frac{(k+1)^2}{1+(k+1)^2}, & k = n^2, \\
\frac{k^2}{1+k^2} - 1, & k = n^2, \\
\frac{k^2}{1+k^2} - \frac{(k+1)^2}{1+(k+1)^2}, & \text{otherwise}.
\end{cases}$$

Therefore, for $\frac{1}{2} < \alpha \leq 1$, we get

$$\frac{1}{n^\alpha} \sum_{k \leq n} |\Delta f_k(x) - 0| \geq \varepsilon, \text{ for every } x \in [0,1];$$

$$\frac{1}{n^\alpha} \sum_{k \leq n} |\Delta f_k(x)| \geq \varepsilon, \text{ for every } x \in [0,1];$$

$$\leq \frac{2\sqrt{n}}{n^{\alpha}} \to 0, \text{ as } n \to \infty.$$

Then $S^\beta - \lim \Delta f_k(x) = 0$, i.e. $x \in S^\beta(\Delta, f)$ for $\frac{1}{2} < \beta \leq 1$, but $x \notin S^\alpha(\Delta, f)$ for $0 < \alpha \leq \frac{1}{2}$.

**Corollary 1.** If a sequence of functions $\{f_k\}$ is $\Delta^n$–statistically convergent of order $\alpha$ to $f$, then it is $\Delta^n$–statistically convergent to $f$.

**Definition 4.** Let $A$ be a subset of $\mathbb{R}$, $\{f_k\}$ be a sequence of real valued functions defined on $A$, and $\alpha$ be any real number such that $0 < \alpha \leq 1$. A sequence of functions $\{f_k\}$ is said to be strongly pointwise $\Delta^n_q$–Cesàro summable of order $\alpha$ for $x \in A$ and $\varepsilon > 0$ if there exists a function $f$ such
that
\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \sum_{k=1}^{n} |\Delta^nf_k(x) - f(x)|^q = 0.
\]
In this case we write \( w_q^\alpha - \lim \Delta^nf_k(x) = f(x) \) on \( A \). The set of all strongly pointwise \( \Delta_q^\alpha \)–Cesàro summable sequences of functions of order \( \alpha \) is denoted by \( w_q^\alpha(\Delta^m,F(p)) \). We write \( w_{\alpha,q}(\Delta^m,F(p)) \) in the case of \( f(x) = 0 \).

**Theorem 4.** Let \( \alpha, \beta \in (0,1] \) with \( \alpha \leq \beta \) and \( 0 < q < \infty \). Then \( w_q^\alpha(\Delta^m,F(p)) \subseteq w_q^\beta(\Delta^m,F(p)) \) and the inclusion is strict for some \( \alpha \) and \( \beta \) such that \( \alpha \leq \beta \).

**Proof.** The inclusion part of the proof is easy, so we omit it. To show the strictness of the inclusion, consider a sequence \( \{ f_k \} \) defined by
\[
f_k(x) = \begin{cases} \frac{kx}{1+nx} & k = n^2 \\ 0 & \text{otherwise} \end{cases}, \]
where \( x \in [1,2] \).

Then we calculate \( \Delta f_k(x) \) as
\[
\Delta f_k(x) = \begin{cases} \frac{kx}{1+nx} & k = n^2 \\ -\frac{kx}{1+(k+1)x} & k = n^2 - 1 \\ 0 & \text{otherwise} \end{cases},
\]
and so
\[
\frac{1}{n^\beta} \sum_{k=1, x \in A}^{n} |\Delta f_k(x) - 0|^p \leq \frac{2\sqrt{n}}{n^\beta} = \frac{2}{n^{\frac{\beta}{2}}}. 
\]
Since \( 2\left( n^{\beta - \frac{1}{2}} \right) \to 0 \) as \( n \to \infty \), then \( w_q^\beta - \lim \Delta f_k(x) = 0 \), which means that the sequence \( \{ f_k \} \) is strongly pointwise \( \Delta_q^- \)–Cesàro summable of order \( \alpha \) for \( \frac{1}{2} < \beta \leq 1 \). But since
\[
\frac{2\sqrt{n}}{2n^\alpha} \leq \frac{1}{n^\alpha} \sum_{k=1, x \in A}^{n} |\Delta f_k(x) - 0|^p
\]
and \( 2\sqrt{n}/2n^\alpha \to \infty \) as \( n \to \infty \), the sequence \( \{ f_k \} \) is not strongly \( \Delta_q^- \)–Cesàro summable of order \( \alpha \) for \( 0 < \alpha < \frac{1}{2} \).

**Corollary 2.** Let \( q \) be a positive real number. Then the inclusion \( w_q^\alpha(\Delta^m,F(p)) \subseteq w_q(\Delta^m,F(p)) \) strictly holds for some \( \alpha \in (0,1] \).

**Theorem 5.** Let \( \alpha, \beta \in (0,1] \) with \( \alpha \leq \beta \) and \( 0 < q < \infty \). If a sequence of functions \( \{ f_k \} \) is strongly pointwise \( \Delta_q^\alpha \)–Cesàro summable of order \( \alpha \) to \( f \), then it is pointwise \( \Delta^m \)–statistically convergent of order \( \beta \) to \( f \).

**Proof.** For any sequence of functions \( \{ f_k \} \) defined on \( A \), we can write
\[
\sum_{k=1, x \in A}^{n} |\Delta^m f_k(x) - f(x)|^q \geq \left\{ \begin{array}{ll}
\sum_{k \leq n} |\Delta^m f_k(x) - f(x)|^q \\
0, & \text{for every } x \in A
\end{array} \right.,
\]
which leads to the consequence that
\[
\frac{1}{n^\alpha} \sum_{k=1, x \in A}^n |\Delta^m f_k(x) - f(x)|^q \geq \frac{1}{n^\alpha} \left( \sum_{k \leq n} \left| \Delta^m f_k(x) - f(x) \right| \geq \varepsilon, \text{ for every } x \in A \right)^q \geq \frac{1}{n^\beta} \left( \sum_{k \leq n} \left| \Delta^m f_k(x) - f(x) \right| \geq \varepsilon, \text{ for every } x \in A \right)^q.
\]

**Corollary 3.** Let \( \alpha \) be a fixed real number such that \( 0 < \alpha \leq 1 \) and \( 0 < q < \infty \). If a sequence of functions \( \{f_k\} \) is strongly pointwise \( \Delta^m \)-Cesàro summable of order \( \alpha \) to \( f \), then it is pointwise \( \Delta^m \)-statistically convergent of order \( \alpha \) to \( f \).

**Definition 5.** Let \( A \) be a subset of \( \mathbb{R} \), \( \{f_k\} \) be a sequence of real valued functions defined on \( A \), and \( \alpha \) be a fixed real number such that \( 0 < \alpha \leq 1 \). Then \( \{f_k\} \) is said to be uniformly and \( \Delta^m \)-statistically convergent of order \( \alpha \) to \( f \) on \( A \) if there exists a natural number \( N = N(\varepsilon) \) for \( x \in A \) and \( \varepsilon > 0 \) such that

\[
\frac{1}{n^\alpha} \sum_{k=1, x \in A}^n |\Delta^m f_k(x) - f(x)|^q < \varepsilon \quad \text{a.a.k}(\alpha) \text{ and for every } x \in A,
\]

meaning that for \( \varepsilon > 0 \),

\[
\lim_{n} \frac{1}{n^\alpha} \left( \sum_{k \leq n} \left| \Delta^m f_k(x) - f(x) \right| \geq \varepsilon \right) = 0 \text{ for every } x \in A.
\]

In this case we write \( S_{n^\alpha}^m - \lim \Delta^m f_k(x) = f(x) \) on \( A \). The set of all uniform, \( \Delta^m \)-statistically convergent sequences of function order \( \alpha \) is denoted by \( S^\alpha(\Delta^m,F(u)) \). In this definition the natural number \( N \) depends only on \( \varepsilon \). Therefore, if a sequence is uniformly and \( \Delta^m \)-statistically convergent to \( f \), then it is pointwise \( \Delta^m \)-statistically convergent to \( f \). However, the converse does not hold in general. To show this, consider a sequence \( \{f_k\} \) defined by

\[
f_k(x) = \begin{cases} 
\frac{2}{kx} & \text{if } \frac{kx}{k+1}^2 \\
2 & \text{otherwise}, \\
\end{cases} \quad n = 1,2,... \text{and } x \in [0,1].
\]

Then

\[
\Delta f_k(x) = \begin{cases} 
\frac{2}{kx} - \frac{(k+1)x}{(k+1)^2} & \text{if } k = n^2 \\
\frac{kx}{(k+1)x} - \frac{2}{(k+1)x} & \text{if } k = n^2 - 1n = 1,2,... \text{and } x \in [0,1]. \\
\end{cases}
\]

Therefore, \( \{f_k\} \) is pointwise \( \Delta \)-statistically convergent of order \( \alpha \) to \( f(x) = 0 \) on \([0,1]\) for \( 0 < \alpha < \frac{1}{2} \), but \( \{f_k\} \) is not uniformly and \( \Delta \)-statistically convergent of order \( \alpha \) in the following theorem since \( \lim_{k \to \infty} c_k \) does not exist, where

\[
c_k = \max_{x \in [0,1]} |\Delta f_k(x) - 0| = \begin{cases} 
\frac{2}{\sqrt{2}} - 2 & \text{if } k = n^2 - 1n = 1,2,..., \\
\frac{2}{\sqrt{A}} & \text{otherwise},
\end{cases}
\]

in which

\[
A = \frac{\sqrt{6k^2+6k}}{2k+5+\sqrt{16k^2+16k+1}} - \frac{\sqrt{6k^2+6k}}{2k+5+\sqrt{16k^2+16k+1}}.
\]

\[\text{Maejo Int. J. Sci. Tec} \]
Theorem 6. Let \( f \) and \( \Delta^\alpha f_k \) (for all \( k \in \mathbb{N} \)) be continuous functions on \( A = [a, b] \subseteq \mathbb{R} \) and \( 0 < \alpha \leq 1 \). Then \( S^\alpha - \lim \Delta^m f_k(x) = f(x) \) uniformly on \( A \) if and only if \( S^\alpha - \lim c_k = 0 \), where
\[
c_k = \max_{x \in A} |\Delta^m f_k(x) - f(x)|.
\]

Proof. Omitted.

It is trivial that if \( \lim \Delta^m f_k(x) = f(x) \) is uniform on \( A \), then \( S^\alpha - \lim \Delta^m f_k(x) = f(x) \) is uniform on \( A \). However, the converse is not true in general. To show this, consider the sequence defined by
\[
f_k(x) = \begin{cases} 1 & k = n^2 \\ x^k & \text{otherwise} \end{cases} \quad k = 1, 2, 3, ..., x \in [0, 1].
\]
So we have
\[
\Delta f_k(x) = \begin{cases} 1 - x^{k+1} & k = n^2 \\ x^k - 1 & k = n^2 - 1 \\ x^k - x^{k+1} & \text{otherwise} \end{cases}.
\]
If \( x \in [0, 1] \) and \( \alpha \in \left[\frac{1}{2}, 1\right] \), then \( \{f_k\} \) is uniformly and \( \Delta \)–statistically convergent of order \( \alpha \) to \( f(x) = 0 \) on \( [0, 1] \) since \( S^\alpha - \lim c_k = 0 \), where
\[
c_k = \max_{x \in [0, 1]} |\Delta f_k(x) - 0| = \begin{cases} 1 & k = n^2 \\ 0 & k = n^2 - 1 \\ \left(\frac{k}{k+1}\right)^{\frac{1}{\alpha}} & \text{otherwise} \end{cases}.
\]
However, \( (\Delta f_k(x)) \) is not uniformly convergent on \( [0, 1] \) since \( \lim_{k \to \infty} c_k \) does not exist.

Corollary 3. (i) \( \lim \Delta^m f_k(x) = f(x) \) uniformly on \( A \) \( \Rightarrow \) \( \lim \Delta^m f_k(x) = f(x) \) on \( A \) \( \Rightarrow \) \( S^\alpha - \lim \Delta^m f_k(x) = f(x) \) pointwise on \( A \).
(ii) \( S^\alpha - \lim \Delta^m f_k(x) = f(x) \) uniformly on \( A \) \( \Rightarrow \) \( S^\alpha - \lim \Delta^m f_k(x) = f(x) \) pointwise on \( A \).
(iii) If \( 0 < \alpha \leq \beta \leq 1 \), then \( S^\alpha \left(\Delta^m, F(u)\right) \subseteq S^\beta \left(\Delta^m, F(u)\right) \).

Definition 6. Let \( A \) be a subset of \( \mathbb{R} \), \( \{f_k\} \) be a sequence of real valued functions defined on \( A \), and \( \alpha \in (0, 1] \). Then \( \{f_k\} \) is said to be uniform \( \Delta^\alpha \)–statistical Cauchy sequence of order \( \alpha \) if there exists a natural number \( N = N(\varepsilon) \) for any \( \varepsilon > 0 \) such that
\[
\left|\Delta^m f_k(x) - \Delta^m f_N(x)\right| < \varepsilon \quad a.a.k(\alpha) \quad \text{and for every } x \in A,
\]
meaning that, for \( \varepsilon > 0 \),
\[
\lim_{n \to \infty} \frac{1}{n^\alpha} \left|k \leq n : \left|\Delta^m f_k(x) - f(x)\right| \geq \varepsilon\right| = 0, \quad \text{for every } x \in A.
\]
In this definition the natural number \( N \) only depends on \( \varepsilon \). By \( S^\alpha \left(\Delta^m, F(p)\right) \), we denote the set of uniform \( \Delta^\alpha \)–statistical Cauchy sequences of functions of order \( \alpha \).

The proofs of the following two theorems are similar to those of Theorems 1 and 2. Thus, we give them without proofs.

Theorem 7. The set \( S^\alpha \left(\Delta^m, F(p)\right) \) is a vector space under the usual operations, which are addition and scalar multiplication of the sequences of functions.
Theorem 8. Let $\alpha \in (0,1]$ and $\{f_k\}$ be a sequence of functions defined on a set $A$. The following statements are equivalent:

(i) $\{f_k\}$ is a uniform, $\Delta^n$ – statistically convergent sequence of order $\alpha$ on $A$;

(ii) $\{f_k\}$ is a uniform, $\Delta^n$ – statistically Cauchy sequence of order $\alpha$ on $A$;

(iii) $\{f_k\}$ is a sequence of functions for which there is a uniformly convergent sequence of functions $\{\Delta^n g_k\}$ such that $\Delta^n f_k(x) = \Delta^n g_k(x) \ a.a.k. \alpha$ for all $x \in A$.

ACKNOWLEDGEMENTS

The authors thank the Management Union of the Scientific Research Projects of Fırat University for its financial support under Grant No. FUBAP FF.12.03.

REFERENCES

16. T. Šalát, “On statistically convergent sequences of real numbers”, Math. Slovaca, 1980, 30, 139-