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Full Paper

The Pell-circulant sequences and their applications

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Abstract: The generalised order-*k* Pell-circulant sequence and the *k*-step Pell-circulant sequence are defined by using the circulant matrix which is obtained from the characteristic polynomial of the generalised order-*k* Pell sequence. Then the relations among the elements of the sequences and generating matrices of the sequences are obtained. Also, the cyclic groups which are generated by the generating matrices and the auxiliary equations of the defined recurrence sequences are considered, and then the orders of these groups are examined. Furthermore, the *k*-step Pell-circulant sequence is extended to groups. Finally, the periods of the *k*-step Pell-circulant sequences in the generalised quaternion group Q_{2^n} are obtained as applications of the results produced.

Keywords: circulant matrix, Pell-circulant sequence, quaternion group

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INTRODUCTION AND PRELIMINARIES

Davis [1] defined the circulant matrix $C_n = \left[c_{ij}\right]_{n\times n}$, associated with the numbers $c_0, c_1, \ldots, c_{n-1}$, as follows:

$$C_n = \begin{bmatrix} c_0 & c_{n-1} & \cdots & c_2 & c_1 \\ c_1 & c_0 & \cdots & c_3 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-2} & c_{n-3} & \cdots & c_0 & c_{n-1} \\ c_{n-1} & c_{n-2} & \cdots & c_1 & c_0 \end{bmatrix}.$$

The $(n-1)$th degree polynomial $P(x) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1}$ is called the associated polynomial of the circulant matrix C_n. More information on the circulant matrix C_n can be found in the literature [2-4].

Kilic and Tasci [5] defined the k sequences of the generalised order-*k* Pell numbers as follows:

for $n > 0$ and $1 \le i \le k$,

$$P_n^i = 2P_{n-1}^i + P_{n-2}^i + \cdots + P_{n-k}^i,$$

with initial conditions

$$P_n^i = \begin{cases} 1 & \text{if } n = 1 - i, \\ 0 & \text{otherwise,} \end{cases} \quad \text{for } 1-k \le n \le 0,$$

where P_n^i is the nth term of the ith sequence. If $k = 2$, the generalised order-*k* Pell sequence, $\{P_n^k\}$, is reduced to the usual Pell sequence, $\{P_n\}$. When $i = k$, P_n^k is called the generalised *k*-Pell number. It is easy to see that the characteristic polynomial of the generalised order-*k* Pell sequence is

$$f(x) = x^k - 2x^{k-1} - x^{k-2} - \cdots - 1.$$

Let the $(n+k)$th term of a sequence be defined recursively by a linear combination of the preceding *k* terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. Kalman [6] derived a number of closed-form formulas for the generalised sequence by the companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then by an inductive argument he obtained that

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}.$$

Many of the numbers obtained by using homogeneous linear recurrence relations and their miscellaneous properties have been studied by several authors [7-17] and the cyclic groups via some special matrices have also been obtained [18-22]. The study of recurrence sequences in groups began earlier by Wall [23], who investigated the ordinary Fibonacci sequences in cyclic groups. Recently, the concept was extended to some special linear recurrence sequences [20-22, 24-30]. In this present paper, the generalised order-*k* Pell-circulant sequence and the *k*-step Pell-circulant sequence are defined by using the circulant matrix which is obtained from the characteristic polynomial of the generalised order-*k* Pell sequence, and then the relations among the elements of the sequences and generating matrices of the sequences are produced. Also, the multiplicative orders of the circulant matrix C_{k+1} and the *k*-step Pell-circulant matrix M_k to modulo *m* are considered, and then the rules for the orders of the cyclic groups are obtained such that these groups are generated by reducing these matrices modulo *m*. Furthermore, the *k*-step Pell-circulant sequences in groups are defined, and then these sequences in finite groups are studied. Finally, the

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periods of the *k*-step Pell-circulant sequences in the generalised quaternion group Q_{2^n} for $n \geq 3$ are obtained.

THE PELL-CIRCULANT SEQUENCES

The circulant matrix for the polynomial $f(x)$ can be written as

$$C_{k+1} = \left[c_{ij}\right]_{(k+1)\times(k+1)} = \begin{cases} 1 \text{ if } \left(i = k+1,\ j=1\right) \text{ and } \left(i+1 = j \text{ such that } 1 \leq i \leq k\right), \\ \text{-2 if } \left(i=k,\ j=1\right), \left(i=k+1,\ j=2\right) \text{ and } \left(i+2=j \text{ such that } 1 \leq i \leq k-1\right), \\ \text{-1 otherwise.} \end{cases}$$

For example, the matrices C_3 and C_5 are

$$C_3 = \begin{bmatrix} -1 & 1 & -2 \\ -2 & -1 & 1 \\ 1 & -2 & -1 \end{bmatrix} \text{ and } C_5 = \begin{bmatrix} -1 & 1 & -2 & -1 & -1 \\ -1 & -1 & 1 & -2 & -1 \\ -1 & -1 & -1 & 1 & -2 \\ -2 & -1 & -1 & -1 & 1 \\ 1 & -2 & -1 & -1 & -1 \end{bmatrix}.$$

The generalised order-*k* Pell-circulant sequence is defined by using the matrices C_{k+1} as follows.

If $k = 2$,

$$x_n = \begin{cases} -2x_{n-1} + x_{n-2} - x_{n-3}, & n \equiv 1 \bmod 3, \\ x_{n-2} - x_{n-3} - 2x_{n-4}, & n \equiv 2 \bmod 3, \\ -x_{n-3} - 2x_{n-4} + x_{n-5}, & n \equiv 0 \bmod 3 \end{cases} \quad \text{for } n > 3,$$

where $x_1 = 0, x_2 = 0$ and $x_3 = 1$.

If $k \geq 3$,

$$x_n = \begin{cases} -x_{n-1} - x_{n-2} - \cdots - x_{n-k+2} - 2x_{n-k+1} + x_{n-k} - x_{n-k-1}, & n \equiv 1 \bmod (k+1), \\ -x_{n-2} - x_{n-3} - \cdots - x_{n-k+2} - 2x_{n-k+1} + x_{n-k} - x_{n-k-1} - x_{n-k-2}, & n \equiv 2 \bmod (k+1), \\ \vdots & \\ -x_{n-k+2} - 2x_{n-k+1} + x_{n-k} - x_{n-k-1} - \cdots - x_{n-2k+2}, & n \equiv k-2 \bmod (k+1), \\ -2x_{n-k+1} + x_{n-k} - x_{n-k-1} - \cdots - x_{n-2k+1}, & n \equiv k-1 \bmod (k+1), \\ x_{n-k} + x_{n-k-1} - \cdots - x_{n-2k+1} - x_{n-2k}, & n \equiv k \bmod (k+1), \\ -x_{n-k-1} - x_{n-k-2} - \cdots - x_{n-2k+1} - x_{n-2k} - x_{n-2k-1}, & n \equiv 0 \bmod (k+1) \end{cases} \quad \text{for } n > k+1,$$

where $x_1 = x_2 = \cdots x_k = 0$ and $x_{k+1} = 1$.

For example, the generalised order-5 Pell-circulant sequence is

$$x_n = \begin{cases} -x_{n-1} - x_{n-2} - x_{n-3} - 2x_{n-4} + x_{n-5} - x_{n-6}, & n \equiv 1 \bmod 6, \\ -x_{n-2} - x_{n-3} - 2x_{n-4} + x_{n-5} + x_{n-6} - x_{n-7}, & n \equiv 2 \bmod 6, \\ -x_{n-3} - 2x_{n-4} + x_{n-5} - x_{n-6} + x_{n-7} - x_{n-8}, & n \equiv 3 \bmod 6, \\ -2x_{n-4} + x_{n-5} - x_{n-6} + x_{n-7} - x_{n-8} - x_{n-9}, & n \equiv 4 \bmod 6, \\ x_{n-5} - x_{n-6} + x_{n-7} - x_{n-8} - x_{n-9} - 2x_{n-10}, & n \equiv 5 \bmod 6, \\ -x_{n-6} + x_{n-7} - x_{n-8} - x_{n-9} - 2x_{n-10} + x_{n-11}, & n \equiv 0 \bmod 6 \end{cases} \quad \text{for } n > 6,$$

where $x_1 = x_2 = \cdots x_5 = 0$ and $x_6 = 1$.

For $n \geq 0$, by an inductive argument, $(C_{k+1})^n$ is obtained as

$$(C_{k+1})^n = \begin{bmatrix} x_{n(k+1)+k+1} & x_{n(k+1)+k} & x_{n(k+1)+k-1} & \cdots & x_{n(k+1)+2} & x_{n(k+1)+1} \\ x_{n(k+1)+1} & x_{n(k+1)+k+1} & x_{n(k+1)+k} & \cdots & x_{n(k+1)+3} & x_{n(k+1)+2} \\ x_{n(k+1)+2} & x_{n(k+1)+1} & x_{n(k+1)+k+1} & \cdots & & x_{n(k+1)+3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ x_{n(k+1)+k-1} & x_{n(k+1)+k-2} & x_{n(k+1)+k-3} & \cdots & x_{n(k+1)+k+1} & x_{n(k+1)+k} \\ x_{n(k+1)+k} & x_{n(k+1)+k-1} & x_{n(k+1)+k-2} & \cdots & x_{n(k+1)+1} & x_{n(k+1)+k+1} \end{bmatrix}. \quad (1)$$

It is easy to see that $(C_{k+1})^n$ is a circulant matrix of order $k+1$ and $P(x) = x_{n(k+1)+k+1} + x_{n(k+1)+1}x + \cdots + x_{n(k+1)+k}x^k$ is the associated polynomial of the matrix $(C_{k+1})^n$.

The *k*-step Pell-circulant sequence is defined by

$$a_{n+k+1} = -2a_{n+k} - a_{n+k-1} - \cdots - a_{n+2} + a_{n+1} \text{ for } n \geq 0, \quad (2)$$

where $a_1 = a_2 = \cdots = a_{k-1} = 0, a_k = 1$ and $k \geq 3$. It is noted that the generating function of the *k*-step Pell-circulant sequence $\{a_n\}$ is

$$g(x) = \frac{x^{k-1}}{-x^k + x^{k-1} + \cdots + x^2 + 2x + 1}.$$

By (2), a companion matrix can be written as

$$M_k = [m_{ij}]_{k \times k} = \begin{bmatrix} -2 & -1 & \cdots & -1 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

The matrix M_k is said to be the *k*-step Pell-circulant matrix. It is clear that

$$\begin{bmatrix} a_{n+k+1} \\ a_{n+k} \\ \vdots \\ a_{n+2} \end{bmatrix} = M_k \begin{bmatrix} a_{n+k} \\ a_{n+k-1} \\ \vdots \\ a_{n+1} \end{bmatrix}.$$

For $n \geq k-1$, by an inductive argument, $(M_k)^n$ may be written as

$$(M_k)^n = \begin{bmatrix} a_{n+k} & a_{n+1} - a_{n+2} - \cdots - a_{n+k-1} & a_{n+2} - a_{n+3} - \cdots - a_{n+k-1} & \cdots & a_{n+k-2} - a_{n+k-1} & a_{n+k-1} \\ a_{n+k-1} & a_n - a_{n+1} - \cdots - a_{n+k-2} & a_{n+1} - a_{n+2} - \cdots - a_{n+k-2} & \cdots & a_{n+k-3} - a_{n+k-2} & a_{n+k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n+1} & a_{n-k+2} - a_{n-k+3} - \cdots - a_n & a_{n-k+3} - a_{n-k+4} - \cdots - a_n & \cdots & a_{n-1} - a_n & a_n \end{bmatrix}. \quad (3)$$

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It can be easily seen that $\det M_k = (-1)^{k+1}$. Now it is well known that the Simpson formula for a recurrence sequence can be obtained from the determinant of its generating matrix. For example, the Simpson formula of 3-step Pell-circulant numbers is

$$(a_{n+1})^3 + (a_{n+2})^2 a_{n-1} + a_{n+3}(a_n)^2 - 2a_{n+2}a_{n+1}a_n - a_{n+3}a_{n+1}a_{n-1} = 1,$$

where $n \geq 2$.

CYCLIC GROUPS VIA MATRICES C_{k+1} AND M_k

For a given matrix $A = [a_{ij}]$ of integers, $A(\text{mod } m)$ means that the entries of A are reduced modulo m. Let $\langle A \rangle_m = \{(A)^n (\text{mod } m) \mid n \geq 0\}$. If $\gcd(\det A, m) = 1$, $\langle A \rangle_m$ is a cyclic group. The cardinal of the set $\langle A \rangle_m$ is denoted by $|\langle A \rangle_m|$. Since $\det M_k = (-1)^{k+1}$, it is clear that the set $\langle M_k \rangle_m$ is a cyclic group for every positive integer m. Similarly, the set $\langle C_{k+1} \rangle_m$ is a cyclic group if $\gcd(\det C_{k+1}, m) = 1$.

Now the cyclic groups which are generated by the matrices C_{k+1} and M_k are considered.

Theorem 1. Let p be a prime and let $\langle G \rangle_{p^\alpha}$ be any of the cyclic groups of $\langle C_{k+1} \rangle_{p^\alpha}$ and $\langle M_k \rangle_{p^\alpha}$ such that $\alpha \in N$. If u is the largest positive integer such that $|\langle G \rangle_p| = |\langle G \rangle_{p^u}|$, then $|\langle G \rangle_{p^v}| = p^{v-u} \cdot |\langle G \rangle_p|$ for every $v \geq u$. In particular, if $|\langle G \rangle_p| \neq |\langle G \rangle_{p^2}|$, then $|\langle G \rangle_{p^v}| = p^{v-1} \cdot |\langle G \rangle_p|$ for every $v \geq 2$.

Proof. Consider the cyclic group $\langle M_k \rangle_{p^\alpha}$. Let $|\langle M_k \rangle_{p^\alpha}|$ be denoted by $h(p^\alpha)$. If $(M_k)^{h(p^{a+1})} \equiv I (\text{mod } p^{a+1})$, then $(M_k)^{h(p^{a+1})} \equiv I (\text{mod } p^a)$ where a is a positive integer and I is a $k \times k$ identity matrix. This means that $h(p^a)$ divides $h(p^{a+1})$. Also, writing $(M_k)^{h(p^a)} = I + (m_{ij}^{(a)} \cdot p^a)$ and using the binomial expansion, $(M_k)^{h(p^a) \cdot p}$ is obtained as

$$(M_k)^{h(p^a) \cdot p} = \left(I + (m_{ij}^{(a)} \cdot p^a)\right)^p = \sum_{i=0}^{p} \binom{p}{i} (m_{ij}^{(a)} \cdot p^a)^i \equiv I (\text{mod } p^{a+1}).$$

This yields that $h(p^{a+1})$ divides $h(p^a) \cdot p$. Thus, $h(p^{a+1}) = h(p^a)$ or $h(p^{a+1}) = h(p^a) \cdot p$. It is clear that $h(p^{a+1}) = h(p^a) \cdot p$ holds if and only if there is an $m_{ij}^{(a)}$ which is not divisible by p. Since u is the largest positive integer such that $h(p) = h(p^u)$, $h(p^u) \neq h(p^{u+1})$. That is, there is an $m_{ij}^{(u+1)}$ which is not divisible by p. Therefore, it is seen that $h(p^{u+1}) \neq h(p^{u+2})$. The proof is completed using an inductive method on u.

The proof for $\langle C_{k+1} \rangle_{p^\alpha}$ is similar. □

Theorem 2. Let $\langle G \rangle_m$ be any cyclic groups of $\langle C_{k+1} \rangle_m$ and $\langle M_k \rangle_m$ and let $m = \prod_{i=1}^{t} p_i^{e_i}, (t \geq 1)$ where p_i's are distinct primes. Then $|\langle G \rangle_m| = \text{lcm}\left[\langle G \rangle_{p_1^{e_1}}, \langle G \rangle_{p_2^{e_2}}, \ldots, \langle G \rangle_{p_t^{e_t}}\right]$.

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Proof. Let us consider the cyclic group $\langle C_{k+1}\rangle_m$; then $\gcd(\det C_{k+1}, m)=1$. Let $\left|\langle C_{k+1}\rangle_{p_i^{e_i}}\right|=\lambda_i$ for $1\le i\le t$ and let $\left|\langle C_{k+1}\rangle_m\right|=\lambda$. Then by (1), it is seen that

$$x_{\lambda_i(k+1)+j}\equiv 0 \bmod p_i^{e_i} \text{ for } 1\le j\le k,$$
$$x_{\lambda_i(k+1)+k+1}\equiv 1 \bmod p_i^{e_i}$$

and

$$x_{\lambda(k+1)+j}\equiv 0 \bmod m \text{ for } 1\le j\le k,$$
$$x_{\lambda(k+1)+k+1}\equiv 1 \bmod m.$$

These imply that $x_{\lambda(k+1)+k+1}=\alpha\cdot x_{\lambda_i(k+1)+j}$, $(t\in N)$ for $1\le j\le k+1$. That is, $(C_{k+1})^{\lambda}$ is of the form $\alpha\cdot(C_{k+1})^{\lambda_i}$ for all values of i. Thus, it is verified that $\left|\langle C_{k+1}\rangle_m\right|$ equals the least common multiple of $\left|\langle C_{k+1}\rangle_{p_i^{e_i}}\right|$'s.

The proof for $\langle M_k\rangle_m$ is similar. □

It is well known that a sequence is periodic if, after certain points, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period *k* if the first *k* elements in the sequence form a repeating subsequence.

Reducing the generalised order-*k* Pell-circulant sequence and the *k*-step Pell-circulant sequence by a modulus m, the repeating sequences $\{x_n(m)\}$ and $\{a_n(m)\}$ are obtained as

$$\{x_n(m)\}=\{x_1(m), x_2(m), x_3(m),\ldots,x_j(m),\ldots\}$$

and

$$\{a_n(m)\}=\{a_1(m), a_2(m), a_3(m),\ldots,a_j(m),\ldots\},$$

where $x_j(m)=x_j(\bmod m)$ and $a_j(m)=a_j(\bmod m)$. They have the same recurrence relation as in the definitions of the generalised order-*k* Pell-circulant sequence and the *k*-step Pell-circulant sequence respectively.

Theorem 3. The sequence $\{a_n(m)\}$ is simply periodic for every positive integer m. Similarly, the sequence $\{x_n(m)\}$ is a simply periodic sequence if $\gcd(\det C_{k+1}, m)=1$.

Proof. Let us consider the sequence $\{a_n(m)\}$. Suppose that $Q=\{(q_1,q_2,\ldots,q_k)\mid q_1,q_2,\ldots,q_k\}$ are integers such that $0\le q_1,q_2,\ldots,q_k\le m-1$; then $|Q|=m^k$. Since there are m^k distinct *k*-tuples of elements of Z_m, at least one of the *k*-tuples appears twice in the sequence $\{a_n(m)\}$. Thus, the subsequence following this *k*-tuple repeats; that is, the sequence $\{a_n(m)\}$ is periodic. So if $a_{i+1}(m)\equiv a_{j+1}(m)$, $a_{i+2}(m)\equiv a_{j+2}(m)$, …, $a_{i+k}(m)\equiv a_{j+k}(m)$ and $i>j$, then $i\equiv j \bmod k$. From the definition, it can be easily derived that

$$a_i(m)\equiv a_j(m),\ a_{i-1}(m)\equiv a_{j-1}(m),\ \ldots,\ a_{i-(j-1)}(m)\equiv a_{j-(j-1)}(m)=a_1(m),$$

and this implies that the sequence $\{a_n(m)\}$ is simply periodic.

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The proof for the sequence $\{x_n(m)\}$ is similar. □

The periods of the sequences $\{a_n(m)\}$ and $\{x_n(m)\}$ are denoted by $l_a^k(m)$ and $l_x^k(m)$ respectively. Then the following useful results from (1) and (3) are obtained respectively.

Corollary 1. **i.** If p is a prime such that $\gcd(\det C_{k+1}, p)=1$, then $l_x^k(p)=(k+1)\cdot\left|\langle C_{k+1}\rangle_p\right|$.

ii. $l_a^k(p)=\left|\langle M_k\rangle_p\right|$ for every prime p. □

Let p be a prime and let

$$A(p^\alpha)=\left\{x^n(\bmod p^\alpha): n\in Z, \alpha\geq 1, x^k=-2x^{k-1}-x^{k-2}-\cdots-x+1, k\geq 3\right\}.$$

Then it is clear that the set $A(p^\alpha)$ is a cyclic group. Now a relationship among the characteristic equation of the *k*-step Pell-circulant sequence and the period $l_a^k(m)$ can be given by the following corollary.

Corollary 2. Let p be a prime and let $\alpha\in N$. Then the cyclic group $A(p^\alpha)$ is isomorphic to the cyclic group $\langle M_k\rangle_{p^\alpha}$. □

THE K-STEP PELL CIRCULANT SEQUENCES IN GROUPS

Let G be a finite *j*-generator group and let X be the subset of $\underbrace{G\times G\times G\times\cdots\times G}_{j}$ such that $(x_1,x_2,\ldots,x_j)\in X$ if and only if G is generated by $x_1,x_2,\ldots,x_j$; $(x_1,x_2,\ldots,x_j)$ is called a generating j-tuple for G.

Definition 1. Let $G=\langle X\rangle$ be a finitely generated group such that $X=\{x_1,x_2,\ldots,x_j\}$. Then the *k*-step Pell-circulant sequence in the group G is defined as follows.

If $j=2$,

$$b_1=x_1, b_2=x_2, b_3=(x_1)^{-1}(x_2)^{-2},\ldots, b_k=(b_1)^{-1}\cdots(b_{k-2})^{-1}(b_{k-1})^{-2}$$

and

$$b_{k+n}=(b_n)(b_{n+1})^{-1}\cdots(b_{n+k-2})^{-1}(b_{n+k-1})^{-2} \text{ for } n\geq 1.$$

If $j\geq 3$,

$$b_1=x_1, b_2=x_2,\ldots, b_i=x_j \quad \text{and} \quad b_{i+n}=\begin{cases}(b_1)^{-1}(b_2)^{-1}\cdots(b_{i+n-2})^{-1}(b_{i+n-1})^{-2} & \text{if } j+n\leq k,\\ (b_{i+n-k})(b_{i+n-k+1})^{-1}\cdots(b_{i+n-2})^{-1}(b_{i+n-1})^{-2} & \text{if } j+n>k\end{cases}$$

for $n\geq 1$.

The *k*-step Pell-circulant sequence of a group generated by $x_1,\ldots,x_j$ is denoted by $PC_k(G;x_1,\ldots,x_j)$.

Theorem 4. Let $G=\langle X\rangle$ be a finite group such that $X=\{x_1,x_2,\ldots,x_j\}$. Then a *k*-step Pell-circulant sequence in G is periodic. In particular, if $k\geq j$, then a *k*-step Pell-circulant sequence in G is simply periodic.

Proof. Suppose that n is the order of G. Since there are n^k distinct triples of elements of G, at least one of the *k*-tuples appears twice in the sequence $PC_k(G;x_1,\ldots,x_j)$. Thus, consider the subsequence following this *k*-tuple. Because of the repeating, the sequence is periodic. Let $k \geq j$. Since the sequence $PC_k\left(G; x_1, \ldots, x_j\right)$ is periodic, there exist natural numbers u and v, with $u \geq v$, such that

$$b_{u+1} = b_{v+1},\ b_{u+2} = b_{v+2}, \ldots, b_{u+k} = b_{v+k}.$$

By the defining relation of the *k*-step Pell-circulant sequence, it is known that

$$\left(b_{k+n}\right)\left(b_{n+k-1}\right)^2\left(b_{n+k-2}\right)\cdots\left(b_{n+1}\right) = \left(b_n\right) \text{ for } j = 2$$

and

$$b_{i+n}\left(b_{i+n-1}\right)^2\left(b_{i+n-2}\right)\cdots\left(b_{i+n-k+1}\right) = \left(b_{i+n-k}\right) \text{ for } j \geq 3.$$

Therefore, $b_u = b_v$, and hence

$$b_{u-1} = b_{v-1}, b_{u-2} = b_{v-2}, \ldots,\ b_{u-(v-1)} = b_{v-(v-1)} = b_1,$$

which implies that the sequence is simply periodic. □

The period of the sequence $PC_k\left(G; x_1, \ldots, x_j\right)$ is denoted by $LPC_k\left(G; x_1, \ldots, x_j\right)$. From the definition it is clear that the period of $PC_k\left(G; x_1, \ldots, x_j\right)$ depends on the chosen generating set and the order in which the assignments of $x_1, x_2, \ldots, x_j$ are made.

Now the periods of the *k*-step Pell-circulant sequences in the generalised quarternion group Q_{2^n} is considered. The generalised quaternion group Q_{2^n}, $\left(n \geq 3\right)$ is defined by the presentation

$$Q_{2^n} = \left\langle x, y : x^{2^{n-1}} = e,\ y^2 = x^{2^{n-2}},\ y^{-1}xy = x^{-1} \right\rangle.$$

Note that $\left|Q_{2^n}\right| = 2^n$, $|x| = 2^{n-1}$ and $|y| = 4$.

Theorem 5. The periods of the *k*-step Pell-circulant sequences in the generalised quaternion group Q_{2^n} for the generating pair $\left(x, y\right)$ are obtained as follows.

i. $LPC_3\left(Q_{2^n}; x, y\right) = 7$.

ii. $LPC_k\left(Q_{2^n}; x, y\right) = 2^{n-2} \cdot l_a^k\left(2\right)$ for $k \geq 4$.

Proof. This is proved by direct calculation. Note that $l_a^3\left(2\right) = 7$.

i. The sequence $PC_3\left(Q_{2^n}; x, y\right)$ is

$$x, y, x^{-1}y^{-2} = x^{n-1}, xy^{-1}\left(x^{n-1}\right)^{-2} = y^{-1}x, y\left(x^{n-1}\right)^{-1}\left(y^{-1}x\right)^{-2} = yx, x^{n-1}\left(y^{-1}x\right)^{-1}\left(yx\right)^{-2} = x^{-2}y,$$
$$y^{-1}x\left(yx\right)^{-1}\left(x^{-2}y\right)^{-2} = e, yx\left(x^{-2}y\right)^{-1} = x, x^{-2}yex^{-2} = y, ex^{-1}y^{-2} = x^{n-1}, \ldots,$$

which has period 7.

ii. If $k \geq 4$, then the sequence $PC_k\left(Q_{2^n}; x, y\right)$ is

$$x_1 = x, x_2 = y, x_3 = x^{n-1}, x_4 = y^{-1}x^3, \cdots,$$
$$x_{2i \cdot l_a^k(2)-k+4} = x^{\lambda_1 4i}, \ldots, x_{2i \cdot l_a^k(2)} = x^{\lambda_{k-3} 4i}, x_{2i \cdot l_a^k(2)+1} = x^{\lambda_{k-2} 4i+1}, x_{2i \cdot l_a^k(2)+2} = x^{\lambda_{k-1} 4i}, x_{2i \cdot l_a^k(2)+3} = x^{n-1}, \ldots,$$

where $\lambda_1, \ldots, \lambda_{k-3}, \lambda_{k-1}$ are positive integers and λ_{k-2} is an odd positive integer such that $\gcd\left(\lambda_1, \ldots, \lambda_{k-3}, \lambda_{k-1}, \lambda_{k-2}\right) = 1$. So the smallest integer i is needed such that $2^{n-1} \mid 4i$ for $n \geq 3$. If

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$i = n-3$, then it is obtained that $x_{2^{n-2}\cdot l_a^k(2)-k+4} = \cdots = x_{2^{n-2}\cdot l_a^k(2)} = e$, $x_{2^{n-2}\cdot l_a^k(2)+1} = x$, $x_{2^{n-2}\cdot l_a^k(2)+2} = y$ and $x_{2^{n-2}\cdot l_a^k(2)+3} = x^{n-1}$. Since the cycle begins again with the $\left(2^{n-2}\cdot l_a^k(2)\right)^{\text{nd}}$ element, $LPC_k\left(Q_{2^n}; x, y\right) = 2^{n-2}\cdot l_a^k(2)$. □

Theorem 6. The periods of the *k*-step Pell-circulant sequences in the generalised quaternion group Q_{2^n} for the generating pair (y, x) are obtained as follows.

i. $LPC_3\left(Q_{2^n}; y, x\right) = 2^{n-3}\cdot 7$.

ii. $LPC_k\left(Q_{2^n}; y, x\right) = 2^{n-2}\cdot l_a^k(2)$ for $k \geq 4$.

The proof is similar to that of Theorem 5 and is omitted.

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