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Full Paper

Newton-Kantorovich method for two-dimensional non-linear singular integral equations

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Abstract: Various properties of a class of non-linear singular integral equations defined on a region in the complex plane were investigated. Newton-Kantorovich method for the approximate solution of equations of such type was applied. Sufficient conditions for the convergence of this method in the Holder space were also given.

Keywords: non-linear singular integral equation, Freshet derivative, Newton-Kantorovich method

INTRODUCTION

The theory of non-linear singular integral equations (NLSIEs) has developed significant importance over the last few years as many engineering problems of applied mechanics and applied mathematics are reduced to the solution of such types of non-linear equations. As in the case of general non-linear operators, the principal approach to the study of these equations is based on various kinds of interaction processes. In each case the basic form of the non-linear operators is naturally taken into account.

It is well known that the solutions to a host of familiar problems of mathematical physics, such as elasticity, plasticity, thermo-elasticity and fluid mechanics, have been reduced to solving equations of the NLSIE type. Owing to the fact that these equations are connected to a wide range of problems of applied character, there is a significant interest in the solution of such NLSIEs. The theory of NLSIEs seems to be particularly complicated if closely linked with the problems of applied mechanics. As is well known, the application area of NLSIEs is outstanding in connection to the theories of elasticity, viscoelasticity, thermo-elasticity, hydrodynamics, fluid mechanics and many other fields outside mathematical physics [1-5].

Recent investigations on this topic have observed that, for many non-linear differential equation systems, the solutions of the Dirichlet boundary-value problems which have partial

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derivatives and are defined in a region can be reduced to solving equations of the NLSIE type [6-9]. The solution of the seismic wave equation − of great importance in elastodynamics − is investigated by reducing it to the solution of NLSIE by using Hilbert transformation [9].

Many problems of applied mechanics are to be reduced to the solution of an NLSIE. This approach involves a non-linear generalisation of the linear singular integral equations of the finite-part type and the multidimensional form, which have been investigated by Ladopoulos [1, 2, 10-12]. Moreover, a non-linear integro-differential equation analysis was proposed by Ladopoulos [13], with applications to some basic problems of orthotropic shallow spherical shell stress analysis. Beyond these applications, Ladopoulos and Zisis [4, 14] have examined the existence and the uniqueness of the solution of the NLSIEs defined in Banach spaces while investigating the application of such types of equations in two-dimensional fluid mechanics.

As it is known, the analytical solutions of NLSIEs can only be found in certain special cases. In the absence of analytical solutions, these types of equations are usually solved by approximation methods. From this point of view, it is important to know how to solve the NLSIEs with approximation methods. Over the past few years, there have been many studies of the approximate solutions of NLSIEs [15-27]. The Newton-Kantorovich method is also frequently used to find the approximate solutions of NLSIEs [28, 29].

Consider the following two-dimensional NLSIE:

$$B(\varphi)(z) \equiv F(z,\varphi(z),T_G f(\cdot,\varphi(\cdot))(z),\Pi_G g(\cdot,\varphi(\cdot))(z)) = 0,\ z \in G, \tag{1}$$

where $f, g : D_0 \to \mathbb{C},\ F : G \to \mathbb{C}$ are known continuous functions in their domains of definition, $\varphi(z)$ is an unknown function, $D_0 = \{(z,\varphi) : z \in \overline{G} = \partial G \cup \overset{\circ}{G}, \varphi \in \mathbb{C}\} = \overline{G} \times \mathbb{C}$, ∂G denotes the boundary of the region $G \subset \mathbb{C}$, $\mathbb{C}$ is the complex plane, $\overset{\circ}{G}$ is a set of interior points of the region G, while its closure $\overline{G} = \partial G \bigcup \overset{\circ}{G}$ and $D = \{(z,\varphi,v,w) : z \in \overline{G},\ \varphi, v, w \in \mathbb{C}\} = \overline{G} \times \mathbb{C}^3$. Moreover,

$$T_G h(\cdot)(z) = \frac{-1}{\pi}\iint_G \frac{h(\zeta)}{\zeta - z} d\xi d\eta,\ \ \Pi_G h(\cdot)(z) = \frac{-1}{\pi}\iint_G \frac{h(\zeta)}{(\zeta - z)^2} d\xi d\eta.$$

The existence and uniqueness of the solution of Eq. (1) was proved by Mustafa and Ardil [23]. The main aim of this paper is to apply the Newton-Kantorovich method to deriving the approximate solution of Eq. (1). To this end, it is shown that the non-linear operator $B(\varphi)$ defined by Eq. (1) is the Freshet differentiable operator. Furthermore, the Freshet derivative of non-linear operator $B(\varphi)$ is calculated and sufficient conditions for the convergence of the Newton-Kantorovich method for the approximate solution of Eq. (1) are given.

PRELIMINARIES

Throughout the paper, if the opposite is not indicated, the set $G \subset \mathbb{C}$ is considered as a bounded and simple connected region in the complex plane as it has already been stated in the introduction section.

If, for every $z_1, z_2 \in \overline{G}$ there exist $H > 0$ and $\alpha \in (0,1)$ numbers such that

$$\left|\varphi(z_1) - \varphi(z_2)\right| \le H \cdot \left|z_1 - z_2\right|^\alpha ,$$

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then it is said that the function $\varphi:\bar{G}\to\mathbb{C}$ satisfies the Holder condition on the set $\bar{G}$ with exponent α. The symbol $H_\alpha(\bar{G})$ will denote the set of all functions that satisfies Holder condition on the set $\bar{G}$ with exponent α.

It is well known that the vector space $(H_\alpha(\bar{G});\|\cdot\|_\alpha)$ is a Banach space with the norm

$$\|\varphi\|_\alpha=\|\varphi\|_{H_\alpha(\bar{G})}\equiv\|\varphi\|_\infty+H(\varphi,\alpha;\bar{G}).$$

Here, the sup-norm satisfies $\|\varphi\|_\infty=\max\{|\varphi(z)|:z\in\bar{G}\}$ and defines

$$H(\varphi,\alpha;\bar{G})=\sup\left\{\frac{|\varphi(z_1)-\varphi(z_2)|}{|z_1-z_2|^\alpha}:z_z,z_2\in\bar{G},\ z_1\neq z_2\right\}.$$

Furthermore, for every $z_k\in\bar{G}$ and $(z_k,\varphi_k)\in D_0$, $(z_k,\varphi_k,v_k,w_k)\in D$ for $k=1,2$, suppose that the scalar $\alpha\in(0,1)$ and the positive numbers $m_1,m_2,n_1,n_2,l_1,l_2,l_3,l_4$ exist such that the following inequalities are satisfied:

$$|f(z_1,\varphi_1)-f(z_2,\varphi_2)|\le m_1\cdot|z_1-z_2|^\alpha+m_2\cdot|\varphi_1-\varphi_2|, \tag{2}$$

$$|g(z_1,\varphi_1)-g(z_2,\varphi_2)|\le n_1\cdot|z_1-z_2|^\alpha+n_2\cdot|\varphi_1-\varphi_2|, \tag{3}$$

$$|F(z_1,\varphi_1,v_1,w_1)-F(z_2,\varphi_2,v_2,w_2)|\le l_1\cdot|z_1-z_2|^\alpha+l_2\cdot|\varphi_1-\varphi_2|+l_3\cdot|v_1-v_2|+l_4\cdot|w_1-w_2|. \tag{4}$$

The symbols $H_{\alpha,1}(m_1,m_2;D_0)$, $H_{\alpha,1}(n_1,n_2;D_0)$ and $H_{\alpha,1,1,1}(l_1,l_2,l_3,l_4;D)$ denote the sets of functions that satisfy the conditions (2), (3) and (4) respectively. The following definition is well known in the literature.

Definition 1 [30]. *Let A be an non-linear operator defined on a set E in a Banach space X. Recall that A is said to be Freshet-differentiable at a point $x_0\in E$ if there exists a bounded linear operator B such that*

$$\lim_{\|h\|\to 0}\frac{\|A(x_0+h)-Ax_0-Bh\|}{\|h\|}=0.$$

Operator B is called the Freshet derivative of operator A at point x_0, denoted by $A'(x_0)$.

In our study the Newton-Kantorovich method for the approximate solution of Eq. (1) is applied. Furthermore, it is proved that the following non-linear operator is Freshet differentiable:

$$B(\varphi)(z)\equiv F(z,\varphi(z),T_Gf(\cdot,\varphi(\cdot))(z),\Pi_Gg(\cdot,\varphi(\cdot))(z)),\ z\in G. \tag{5}$$

Let $B'(\varphi)$ be the Freshet differential of non-linear operator $B(\varphi)$. Assume that there exists a solution for the linear equation

$$B'(\varphi)h(z)=\phi(z),\ z\in G, \tag{6}$$

for every $\phi\in H_\alpha(\bar{G}),0<\alpha<1$. This means that the existence of the bounded linear inverse operator $[B'(\varphi)]^{-1}$ is assumed.

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NEWTON-KANTOROVICH METHOD FOR EQUATION (1)

Let X and Y be Banach spaces and $L(X,Y)$ denote the linear operator spaces from X to Y. If $kerA$ and $CokerA = Y/\operatorname{Im} A$ are finite-dimensional, $A \in L(X,Y)$ is called a Fredholm operator. The index of operator A is defined by $\kappa = indA = \dim\ker A - \dim Co\ker A$. The family of the Fredholm transformations from X to Y with index κ is denoted by $\phi_\kappa(X,Y)$.

Let $U \subset X$ be an open set and $h: U \to Y$ be a transformation. If $h'(\varphi) \in \phi_\kappa(X,Y)$ for every $\varphi \in X$, then the transformation $h: U \to Y$ is called a Fredholm operator with index κ from class C'. Here, $h'(\varphi)$ is a Freshet differential of operator $h: U \to Y$.

In this study the family of Fredholm transformations C' from U to Y with index κ is denoted by $\phi_\kappa C'(U,Y)$.

Now the following theorem about the existence of a Freshet derivative for non-linear operator $B(\varphi)$ is presented.

Theorem 1. *Let functions* $F(t,u,v,w)$, $f(t,u)$, $g(t,u)$ *and derivatives* $F_u^{'}, F_v^{'}, F_w^{'}, F_{u^2}^{''}, F_{v^2}^{''}, F_{w^2}^{''}, F_{uv}^{''}, F_{uw}^{''}, F_{vw}^{''}$ *and* $f_u^{'}, f_{u^2}^{''}, g_u^{'}, g_{u^2}^{''}$ *be of class* $H_{\alpha,1,1,1}(l_1,l_2,l_3,l_4;D)$, $H_{\alpha,1}(m_1,m_2;D_0)$ *and* $H_{\alpha,1}(n_1,n_2;D_0)$, $0<\beta<\alpha\le 1$ *respectively. Then non-linear operator* $B(\varphi)$ *defined by (5) is Freshet differentiable for every* $\varphi \in H_\beta(\bar{G})$ *, and the derivative can be written:*

$$\begin{aligned} B'(\varphi)h(z) = {} & F_u^{'}(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z),\Pi_G g(\tau,\varphi(\tau))(z)) \cdot h(z) \\ & + F_v^{'}(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z),\Pi_G g(\tau,\varphi(\tau))(z)) \cdot T_G(f_u^{'}(\tau,\varphi(\tau))h(\tau)) \\ & + F_W^{'}(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z),\Pi_G g(\tau,\varphi(\tau))(z)) \cdot \Pi_G(g_u^{'}(\tau,\varphi(\tau))h(\tau)). \end{aligned} \tag{7}$$

Furthermore, Freshet derivative $B'(\varphi)$ *on the ball* $U(\varphi_0,r) = \{\varphi \in H_\beta(\bar{G}) : \|\varphi_0 - \varphi\| \le r\}$ *provides the following Lipchitz condition:*

$$\|B'(\varphi_1) - B'(\varphi_2)\| \le L \cdot \|\varphi_1 - \varphi_2\|, \ \varphi_1, \varphi_2 \in U(\varphi_0, r), \tag{8}$$

where L *is a constant and depends on functions* F, f, g *and* $r, \varphi_0 \in H_\beta(\bar{G})$.

Proof. Firstly, to prove that the equality (7) is correct, let functions $\varphi, h \in H_\beta(\bar{G})$ and $\beta \in (0,1)$ be given.

Now

$$\begin{aligned} B(\varphi+h) - B(\varphi) = {} & F(z,(\varphi+h)(z),T_G f(z,(\varphi+h)(\tau))(z),\Pi_G g(z,(\varphi+h)(\tau))(z)) \\ & - F(z,\varphi(z),T_G f(z,\varphi(\tau))(z),\Pi_G g(z,\varphi(\tau))(z)). \end{aligned}$$

Then

$$\begin{aligned} B(\varphi+h) - B(\varphi) = {} & [F(z,(\varphi+h)(z),T_G f(z,(\varphi+h)(\tau))(z),\Pi_G g(z,(\varphi+h)(\tau))(z)) \\ & - F(z,\varphi(z),T_G f(z,(\varphi+h)(\tau))(z),\Pi_G g(z,(\varphi+h)(\tau))(z))] \\ & + [F(z,\varphi(z),T_G f(z,(\varphi+h)(\tau))(z),\Pi_G g(z,(\varphi+h)(\tau))(z)) \\ & - F(z,\varphi(z),T_G f(z,\varphi(\tau))(z),\Pi_G g(z,(\varphi+h)(\tau))(z))] \\ & + [F(z,\varphi(z),T_G f(z,\varphi(\tau))(z),\Pi_G g(z,(\varphi+h)(\tau))(z)) \\ & - F(z,\varphi(z),T_G f(z,\varphi(\tau))(z),\Pi_G g(z,\varphi(\tau))(z))]. \end{aligned}$$

From the assumptions of the theorem, the following can be written:

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$$
\begin{aligned}
B(\varphi+h)-B(\varphi) &= F_u^{'}(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z),\Pi_G g(\tau,\varphi(\tau))z))\cdot h(z)\\
&+F_v^{'}(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z),\Pi_G g(\tau,\varphi(\tau))z))\cdot T_G(f_u^{'}(\tau,\varphi(\tau))\cdot h(\tau))\\
&+F_w^{'}(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z),\Pi_G g(\tau,\varphi(\tau))z))\cdot \Pi_G(g_u^{'}(\tau,\varphi(\tau))\cdot h(\tau))+\omega(\varphi,h)(z)
\end{aligned}
\tag{9}
$$

Here,

$$\omega(\varphi,h)(z)=\omega_1(\varphi,h)(z)+\omega_2(\varphi,h)(z)+\omega_3(\varphi,h)(z), \tag{10}$$

$$\omega_1(\varphi,h)(z)=\int_0^1\left[\begin{array}{l}F_u^{'}(z,(\varphi+\theta\cdot h)(z),T_G f(\tau,(\varphi+\theta\cdot h)(\tau))(z)),\Pi_G g(\tau,(\varphi+\theta\cdot h)(z)))-\\ -F_u^{'}(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z)),\Pi_G g(\tau,\varphi(z)))\end{array}\right]\cdot h(z)d\theta, \tag{11}$$

$$
\begin{aligned}
\omega_2(\varphi,h)(z)=\int_0^1&\left[\begin{array}{l}F_v^{'}(z,\varphi(z),T_G f(\tau,(\varphi+\theta\cdot h)(\tau))(z)),\Pi_G g(\tau,(\varphi+\theta\cdot h)(z)))-\\ -F_v^{'}(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z)),\Pi_G g(\tau,\varphi(z)))\end{array}\right]\\
&\times T_G(f_u^{'}(\tau,\varphi(\tau))(z)h(z)d\theta,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\omega_3(\varphi,h)(z)=\int_0^1&\left[\begin{array}{l}F_w^{'}(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z)),\Pi_G g(\tau,(\varphi+\theta\cdot h)(z)))\\ -F_v^{'}(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z)),\Pi_G g(\tau,\varphi(z)))\end{array}\right]\\
&\times \Pi_G(g_u^{'}(\tau,\varphi(\tau))(z)h(z)d\theta.
\end{aligned}
\tag{13}
$$

Let $D_r=\{(z,\varphi,v,w):\ z\in\bar{G},\|\varphi-\varphi_0(\tau)\|\le r, v,w\in\mathbb{C}\},\ r>0$. From the assumptions of the theorem and properties of operators T_G and Π_G , it can be seen that derivative $F_u^{'}(z,u,v,w)$ is uniformly continuous on D_r. Therefore, for any $\varepsilon>0,$ the following evaluation can be written:

$$
\begin{aligned}
\|\omega_1(\varphi,h)\|_\infty &\le \max\left\{\begin{array}{l}\int_0^1\left|\begin{array}{l}F_u^{'}(z,(\varphi+\theta\cdot h)(z),T_G f(\tau,(\varphi+\theta\cdot h)(\tau))(z)),\Pi_G g(\tau,(\varphi+\theta\cdot h)(\tau))(z))\\ -F_u^{'}(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z)),\Pi_G g(\tau,\varphi(\tau)(z))\end{array}\right|\\ \times|h(z)|d\theta : z\in\bar{G}\end{array}\right\}\\
&\le c_1\cdot\varepsilon\cdot\|h\|_\infty .
\end{aligned}
$$

It follows that

$$\lim_{\|h\|\to 0}\frac{\|\omega_1(\varphi,h)\|}{\|h\|}=0. \tag{14}$$

The following limits can be proved in a manner similar to (14):

$$\lim_{\|h\|\to 0}\frac{\|\omega_2(\varphi,h)\|}{\|h\|}=0 \text{ and } \lim_{\|h\|\to 0}\frac{\|\omega_3(\varphi,h)\|}{\|h\|}=0. \tag{15}$$

From (14) and (15), the following is obtained:

$$\lim_{\|h\|\to 0}\frac{\|\omega\ (\varphi,h)\|}{\|h\|}=0.$$

Hence by the definition of the Freshet derivative, the veracity of the equality (7) is proved.

Now to prove that Freshet derivative $B'(\varphi)$ provides Lipchitz condition (8) on ball $U(\varphi_0,r),$ let $\varphi_1,\varphi_2\in U(\varphi_0,r).$ Then $(B'(\varphi_1)-B'(\varphi_2))h(z)$ is obtained as:

$$\begin{aligned}(B'(\varphi_1)-B'(\varphi_2))h(z)&=[F_u'(z,\varphi_1(z),T_G f(\tau,\varphi_1(\tau))(z),\Pi_G g(\tau,\varphi_1(\tau))(z))\\&-F_u'(z,\varphi_2(z),T_G f(\tau,\varphi_2(\tau))(z),\Pi_G g(\tau,\varphi_2(\tau))(z))]\cdot h(z)\\&+[F_v'(z,\varphi_1(z),T_G f(\tau,\varphi_1(\tau))(z),\Pi_G g(\tau,\varphi_1(\tau))(z))\cdot T_g(f_u'(\xi,\varphi_1(\xi))(\tau)h(\tau))\\&-F_v'(z,\varphi_2(z),T_G f(\tau,\varphi_2(\tau))(z),\Pi_G g(\tau,\varphi_2(\tau))(z))\cdot T_g(f_u'(\xi,\varphi_2(\xi))(\tau)h(\tau))]\\&+[F_w'(z,\varphi_1(z),T_G f(\tau,\varphi_1(\tau))(z),\Pi_G g(\tau,\varphi_1(\tau))(z))\cdot\Pi_g(g_u'(\xi,\varphi_1(\xi))(\tau)h(\tau))\\&-F_v'(z,\varphi_2(z),T_G f(\tau,\varphi_2(\tau))(z),\Pi_G g(\tau,\varphi_2(\tau))(z))\cdot\Pi_g(g_u'(\xi,\varphi_2(\xi))(\tau)h(\tau))].\end{aligned}\tag{16}$$

From the assumptions of the theorem and properties of operators T_G and Π_G, it is seen that

$$\left\|F_u'(\cdot,\varphi_1(\cdot),T_G f(\cdot,\varphi_1(\cdot))(\cdot),\Pi_G g(\cdot,\varphi_1(\cdot))(\cdot))-F_u'(\cdot,\varphi_2(\cdot),T_G f(\cdot,\varphi_2(\cdot))(\cdot),\Pi_G g(\cdot,\varphi_2(\cdot))(\cdot))\right\|\le L_1\cdot\|\varphi_1-\varphi_2\|.\tag{17}$$

Similar evaluations can be proved for the second and third terms of difference (16). From these evaluations, it is seen that condition (8) is true. Thus, the proof of Theorem 1 is complete.

The Freshet derivative in the form of a linear singular integral operator can be written as follows:

$$B'(\varphi)h(z)=a(\varphi,z)\cdot h(z)+b(\varphi,z)\cdot T_G(f_u'(\tau,\varphi(\tau))h(\tau))+c(\varphi,z)\cdot\Pi_G(g_u'(\tau,\varphi(\tau))h(\tau)).\tag{18}$$

Here,

$$a(\varphi,z)=F_u'(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z),\Pi_G g(\tau,\varphi(\tau))(z)),$$

$$b(\varphi,z)=F_v'(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z),\Pi_G g(\tau,\varphi(\tau))(z)),$$

$$c(\varphi,z)=F_W'(z,\varphi(z),T_G f(\tau,\varphi(\tau))(z),\Pi_G g(\tau,\varphi(\tau))(z)).$$

The existence of the only zero solution of the equation below in space $H_\beta(\bar{G})$ is assumed:

$$B'(\varphi)h(z)=0.\tag{19}$$

In this case the following equation is the unique solution for every $\phi\in H_\beta(\bar{G})$:

$$B'(\varphi)h(z)=\phi(z).\tag{20}$$

Therefore, bounded linear inverse operator $\left[B'(\varphi_0)\right]^{-1}:H_\beta(\bar{G})\to H_\beta(\bar{G})$ exists. As a result, the solution of Eq. (20) is given as:

$$h(z)=\left[B'(\varphi_0)\right]^{-1}\phi(z).$$

Now a theorem on convergence of the Newton-Kantorovich method for Eq. (1) is given.

Theorem 2. *Let the conditions of Theorem 1 be provided and* $\kappa=indB'(\varphi_0)\ge 0$ *for a* $\varphi_0\in H_\beta(\bar{G})$. *Furthermore, assume that homogeneous Eq. (19) has only a trivial solution. Also, suppose that*

$$\left\|\left[B'(\varphi_0)\right]^{-1}\right\|\le m,\ \left\|\left[B'(\varphi_0)\right]^{-1}\cdot B(\varphi_0)\right\|\le M.$$

If

$$\delta=LMm<\frac{1}{2}\ \ and\ \ r\ge r_0=\frac{1-\sqrt{1-2\delta}}{\delta}\cdot M,$$

then the following equation has a unique solution φ^*, *which is in ball* $U(\varphi_0,r_0)=\left\{\varphi\in H_\beta(\bar{G}):\ \|\varphi_0-\varphi\|\le r_0\right\}$:

$$B(\varphi)(z)=0,\ z\in G.\tag{21}$$

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Furthermore, the following successive approximations converge to the solution φ^* *of Eq. (21) in the ball* $U(\varphi_0, r_0)$:

$$\varphi_{n+1}(z) = \varphi_n(z) - \left[B'(\varphi_0)\right]^{-1} \cdot B(\varphi_n),\ n = 0,1,\dots\ . \tag{22}$$

Also, the convergence ratio is to be taken as the following:

$$\left\|\varphi^* - \varphi_n\right\| \le \frac{(1-\sqrt{1-2\delta})^n}{\sqrt{1-2\delta}} \cdot M,\ \ n = 0,1,\dots\ .$$

Proof. Let $\varphi_0 \in H_\beta(\bar{G}),\ r_0 = \dfrac{1-\sqrt{1-2\delta}}{\delta} \cdot M$ and $U(\varphi_0, r_0) = \left\{\varphi \in H_\beta(\bar{G}):\ \left\|\varphi_0 - \varphi\right\| \le r_0\right\}$. It must then be proved that the Newton-Kantorovich method is applied to the approximate solution of Eq. (21).

If

$$A(\varphi)(z) = \varphi(z) - \left[B'(\varphi_0)\right]^{-1} B(\varphi)(z),$$

then Eq. (21) can be written as

$$\varphi(\mathrm{z}) = A(\varphi)(\mathrm{z}). \tag{23}$$

In this case applying the Newton-Kantorovich method to the approximate solution of Eq. (21) is equivalent to applying the iteration method to the approximate solution of Eq. (23).

Now let us show that the iteration method to the approximate solution of Eq. (23) can be applied. To this end, it is sufficient to show that operator A satisfies the contraction mapping principle conditions.

Now it is to be shown that operator A maps from ball $U(\varphi_0, r_0)$ to itself and that it is a contraction mapping. It can be written that for every $\varphi \in U(\varphi_0, r_0)$,

$$\int_0^1 B'(\varphi_0 + \theta(\varphi - \varphi_0))(\varphi - \varphi_0)d\theta = \int_0^1 B'(\varphi_0 + \theta(\varphi - \varphi_0))d(\theta(\varphi - \varphi_0)) = \int_{\varphi_0}^{\varphi} B'(t)dt = B(\varphi) - B(\varphi_0).$$

Thus, the following equality is true:

$$B(\varphi) - B(\varphi_0) = \int_0^1 B'(\varphi_0 + \theta(\varphi - \varphi_0))(\varphi - \varphi_0)d\theta. \tag{24}$$

Let $\varphi_1, \varphi_2 \in U(\varphi_0, r_0)$. Using Eq. (24), it can be written that

$$B(\varphi_1) - B(\varphi_2) - B'(\varphi_2)(\varphi_1 - \varphi_2) = \int_0^1 \left[B'(\varphi_2 + \theta(\varphi_1 - \varphi_2)) - B'(\varphi_2)\right] d\theta(\varphi_1 - \varphi_2).$$

This gives the following:

$$\left\|B(\varphi_1) - B(\varphi_2) - B'(\varphi_2)(\varphi_1 - \varphi_2)\right\| \le L\int_0^1 \left\|\varphi_1 - \varphi_2\right\|\left\|\varphi_1 - \varphi_2\right\|\theta d\theta = \frac{L}{2}\left\|\varphi_1 - \varphi_2\right\|^2.$$

Therefore,

$$\left\|B(\varphi_1) - B(\varphi_2) - B'(\varphi_2)(\varphi_1 - \varphi_2)\right\| \le \frac{L}{2}\left\|\varphi_1 - \varphi_2\right\|^2. \tag{25}$$

Now let us show that operator A maps from ball $U(\varphi_0, r_0)$ to itself. The following inequality is clear:

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$$\|A(\varphi)-\varphi_0\| \le \|A(\varphi)-A(\varphi_0)\| + \|A(\varphi_0)-\varphi_0\| = \left\|\left[B'(\varphi_0)\right]^{-1}\left[B(\varphi)-B(\varphi_0)-B'(\varphi_0)(\varphi-\varphi_0)\right]\right\|$$
$$+\left\|\left[B'(\varphi_0)\right]^{-1}B(\varphi_0)\right\| \le m\|B(\varphi)-B(\varphi_0)-B'(\varphi_0)(\varphi-\varphi_0)\|+M,\ \varphi\in U(\varphi_0,r_0).$$

Using Eq. (25), it is written that

$$\|A(\varphi)-\varphi_0\| \le \frac{1}{2}mLr_0^2+M.$$

Furthermore, taking $mLr_0^2-2r_0+2M=0$ from the previous inequality, it is obtained that

$$\|A(\varphi)-\varphi_0\| \le r_0.$$

Hence $A(\varphi)\in U(\varphi_0,r_0)$.

Now let us show that operator A is a contraction mapping. Using Eq. (24) it can be written that

$$A(\varphi_1)-A(\varphi_2)=\varphi_1-\varphi_2-\left[B'(\varphi_0)\right]^{-1}\left[B(\varphi_1)-B(\varphi_2)\right]$$
$$=\left[B'(\varphi_0)\right]^{-1}\left[B'(\varphi_0)(\varphi_1-\varphi_2)-B(\varphi_1)+B(\varphi_2)\right]=\left[B'(\varphi_0)\right]^{-1}\int_0^1\left[B'(\varphi_0)-B'(\varphi_2+\theta(\varphi_1-\varphi_2))\right]d(\theta(\varphi_1-\varphi_2)).$$

Thus, it is obtained that

$$\|A(\varphi_1)-A(\varphi_2)\| \le mLr_0\|\varphi_1-\varphi_2\|.$$

Now since $mLr_0<1$, operator A is a contraction mapping with coefficient $q=1-\sqrt{1-2\delta}$. Therefore, according to the contraction mapping principle, Eq. (23) has a unique solution φ^* in ball $U(\varphi_0,r_0)$ and this solution is the limit of the following iteration:

$$\varphi_{n+1}(z)=A(\varphi_n)(z),\ n=0,1,...$$

and

$$\|\varphi_n-\varphi^*\| \le \frac{q^n}{1-q}\|\varphi_1-\varphi_0\|.$$

Also, under the hypothesis of the theorem,

$$\|\varphi_1-\varphi_0\|=\|A(\varphi_0)-\varphi_0\|=\left\|\left[B'(\varphi_0)\right]^{-1}B(\varphi_0)\right\|\le M.$$

Therefore, the proof of Theorem 2 is complete.

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