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Short Communication

Dynamics of thin shell in Vaidya-Reissner-Nordstrom space-time

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Abstract: The general relativistic equations of motion of a Vaidya-Reissner-Nordstrom thin shell are formulated through cut-and- paste technique (Darmois-Israel formalism). These equations of motion are reduced to the standard Ostriker-Gunn equations in the non-relativistic limit.

Keywords: Vaidya-Reissner-Nordstrom space-time, equation of motion, general relativity, shell dynamics, Newton limits

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INTRODUCTION

The dynamics of a shell separating two backgrounds in the context of general relativity has been developed in a powerful and direct formalism since the pioneering work of Israel [1] and applied to the charged shell by Kuchar [2]. Berezin et al. [3] have studied the dynamics of the bubbles in general relativity. Sato [4] investigated the dynamics of the border between two given spaces. Nunez and de Oliveira [5] have discussed the dynamics of massive shell ejected in a supernova explosion.

Ostriker and Gunn [6] have investigated the dynamics of a supernova remnant as a thin shell moving in a background which has a radiating mass inside and dust outside. They started with the Newton's second law for describing the acceleration of the shell; the force terms were introduced essentially by hand while the equation of motion is given by

$$M_s \frac{\mathrm{d}^2 R}{dt^2} = -\mathrm{G}\frac{(M_N + \frac{1}{2}M_s)}{\mathrm{R}^2} M_s + 4\pi R^2 (P_c - P_{is}), \tag{1}$$

where M_s is the mass of the shell, M_N is the mass of the neutron star, R is the radius of the shell, G is the gravitational constant, P_c is the pressure in the cavity related to the radiation emitted by the star, and P_{is} is the pressure due to the contact with the interstellar medium, defined as

$$P_{is} = \rho_{is}\ (\tfrac{dR}{dt})^2, \tag{2}$$

a friction term known as the 'snow blow' effect [7].

The aim of this work is to perform a relativistic study for Ostriker's equation (1). The dynamics of thin shell with a Vaidya-Reissner-Nordstrom metric inside and a Friedman-Robertson metric outside are derived. The results obtained in the present work and by both Ostriker and Gunn [6] and Nunez and de Oliveira [5] equations are analysed.

DYNAMICS OF THIN SHELL

The Vaidya-Reissner-Nordstrom space-time inside the shell is described by Joshi [8]:

$$ds_-^2 = Bdv(-Bf_- dv + 2bdr_-) + r_-^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3}$$

where $f_- = 1 - \frac{2m_-(v)}{R} + \frac{q^2}{R^2}$, $B = e^{\psi}$, in which f_- and ψ are, in general, functions of the null coordinate v and of the radius r, with $m_-(v)$ being the mass function in the interior space, and q is the charge. The Friedman Robertson-Walker dust universe space-time outside the shell is described by Joshi [8]:

$$ds_+^2 = -dt_+^2 + a^2(t)\frac{dl^2}{(1-Kl^2)} + r_+^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4}$$

where $r_+ = a(t(\tau))\, l(\tau)$ is the radius of the shell, $a(\tau)$ is a dimensionless scale factor depending on time τ, called the radius of the universe and $K = 0, \pm 1$ is the curvature constant.

Let the equation of the shell be $r = R(\tau)$, with the function $R(\tau)$ describing the time evolution of the shell. The intrinsic metric on Σ is written as

$$ds^2 = -d\tau^2 + R^2(\tau)(d\theta^2 + \sin^2\theta d\phi^2), \tag{5}$$

where τ is the proper time on the shell. By applying Darmois-Israel formalism [1] to the matter at the hypersurface Σ, the extrinsic curvature associated with the two sides of the shell is defined as

$$K_{ij}^{\pm} = -n_\gamma^{\pm}\left(\frac{\partial^2 x^\gamma}{\partial\xi^i\partial\xi^j} + \Gamma_{\alpha\beta}^{\gamma}\frac{\partial x^\alpha}{\partial\xi^i}\frac{\partial x^\beta}{\partial\xi^j}\right)\Big|_\Sigma \tag{6}$$

where $n_\gamma^{\pm}$ is the unit normal 4-vector. The Einstein equation, which determines the relations between the three-dimensional intrinsic energy momentum tensor and the extrinsic curvature, is given by the Lanczos equations:

$$t_{ij} = \frac{-1}{8\pi}\left(\left[K_{ij}\right] - \left[K\right]g_{ij}\right), \tag{7}$$

where $[K]$ is the trace of $[K_{ij}] = K_{ij}^+ - K_{ij}^-$ and t_{ij} : ($t_j^i = g^{ia}t_{aj}$) is the surface stress-energy tensor on the hypersurface Σ, where $t_j^i = diag(-\sigma, p, p)$, and σ and p are the surface-energy density and pressure respectively [9].

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For a spherically symmetric thin shell, the Lanczos equations are then reduced to

$$\sigma = \frac{-1}{4\pi}\left[K_{\theta}^{\theta}\right], \tag{8}$$

$$p = p_{\phi} = p_{z} = \frac{1}{8\pi}\left(\left[K_{\tau}^{\tau}\right]+\left[K_{\theta}^{\theta}\right]\right) . \tag{9}$$

The equations for the extrinsic curvature $K^{\pm}{}_{ab}$ in M^{+} and M^{-} are

$$K^{-}{}_{\tau\tau} = -\frac{\ddot{\nu}}{\dot{\nu}} - \tfrac{1}{2}\dot{\nu}\, Bf_{,r} - \frac{\dot{B}}{B} - \left(\frac{1+B\dot{R}\,\dot{\upsilon}}{B^{2}\,\dot{\upsilon}}\right)B_{,r}, \tag{10}$$

$$K^{-}{}_{\theta\theta} = R\,(Bf\,\dot{\upsilon} - \dot{R}), \tag{11}$$

$$K^{-}{}_{\phi\phi} = K^{-}{}_{\theta\theta}\,\sin^{2}\theta, \tag{12}$$

and

$$K^{+}{}_{\tau\tau} = \frac{A}{\sqrt{1-KL^{2}}}\left(\dot{L}\left(\ddot{t}_{+} + \frac{A\dot{L}^{2}}{1-KL^{2}}\frac{dA}{dt}\right) - \dot{t}_{+}\left(\ddot{L} + \frac{kL\dot{L}^{2}}{1-KL^{2}} + \frac{2}{A}\dot{L}\,\dot{t}_{+}\frac{dA}{dt_{+}}\right)\right), \tag{13}$$

$$K^{+}{}_{\theta\theta} = \frac{A}{\sqrt{1-KL^{2}}}\left(\dot{L}R^{2}H^{2} + \dot{t}_{+}L(1-KL^{2})\right), \tag{14}$$

$$K^{+}{}_{\phi\phi} = K^{+}{}_{\theta\theta}\,\sin^{2}\theta, \tag{15}$$

where $H = \frac{1}{A}\frac{dA}{dt_{+}}$ and $al = AL$, with the dot representing the derivative with respect to the proper time. The conditions

$$\dot{\upsilon} = \frac{1}{Bf_{-}}\left(\dot{R} \pm \sqrt{f_{-} + \dot{R}^{2}}\right) \tag{16}$$

$$\dot{t}_{+} = \frac{1}{(1-KL^{2}-R^{2}H^{2})}\left(-RH\dot{R} \pm \sqrt{(1-KL^{2})(1+\dot{R}^{2}-KL^{2}-R^{2}H^{2})}\right), \tag{17}$$

must be fulfilled because τ is the proper time on the shell. Equation (11) can be written in the form:

$$K^{-}{}_{\theta\theta} = R(\tau)\; F_{-}, \tag{18}$$

where

$$F_{-} = \sqrt{f_{-} + \dot{R}^{2}} . \tag{19}$$

Similarly, equation (14) becomes

$$K^{+}{}_{\theta\theta} = AL\sqrt{1-KL^{2}} \tag{20}$$

when $\dot{t}_{+} = 1$. Therefore, equation (20) can be written in the form

$$K^{+}{}_{\theta\theta} = R(\tau)\,F_{+}, \tag{21}$$

where

$$F_{+} = \sqrt{f_{+} + \dot{R}^{2}} , \tag{22}$$

with $f_{+} = 1 - \frac{2m_{+}}{R}$ and $m_{+} = \frac{4\pi}{3}\sigma R^{3}$ represents the gravitational mass in the space-time outside the shell.

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The Lanczos equations (8, 9), with equations (10-15), can be written in the form

$$4\pi R\,\sigma = \mathrm{F}_{-} - \mathrm{F}_{+}, \tag{23}$$

$$8\,\pi R^2\,\mathrm{P} = [\mathrm{K}_{\theta\theta}] - R^2[\mathrm{K}_{\tau\tau}]. \tag{24}$$

Equations (19) and (22) can be written in the form

$$\mathrm{F}_{+}^2 - \mathrm{F}_{-}^2 = \frac{-2m}{R} - \frac{q^2}{R^2}, \tag{25}$$

$$\mathrm{F}_{+}\mathrm{F}_{-} = \frac{m^2}{M^2} - \frac{M^2}{4R^2} + \frac{q^2}{2R^2} - \frac{q^4}{4M^2R^2}, \tag{26}$$

with $m = m_{+} - m_{-}$, and $M = 4\pi R^2\sigma$ is the surface mass of the shell. The expression of F on both sides of the shell is given by

$$\mathrm{F}_{\pm} = \frac{m}{M} \mp \frac{M}{2R} \pm \frac{q^2}{2MR}. \tag{27}$$

Rearranging equation (23), the equation of motion of the thin shell becomes

$$\sqrt{\mathrm{f}_{-} + \dot{R}^2} - \sqrt{\mathrm{f}_{+} + \dot{R}^2} = \frac{M}{R}. \tag{28}$$

The radial equation of motion (28) can be written in the form

$$\dot{R}^2 = (\frac{m}{M})^2 - 1 + \frac{m_{+} + m_{-}}{R} + \frac{M^2}{4R^2} - \frac{q^2}{2R^2} + \frac{mq^2}{\mathrm{R}M^2} + \frac{q^4}{4M^2R^2}. \tag{29}$$

This equation represents the energy equation of the shell and can be called the expansion law of the shell. The dynamic equation of motion is

$$\dot{R}^2 + V(R) = 0\ , \tag{30}$$

with

$$V(R) = 1 - (\frac{m}{M})^2 - \frac{m_{+} + m_{-}}{R} - \frac{M^2}{4R^2} + \frac{q^2}{2R^2} - \frac{mq^2}{\mathrm{R}M^2} - \frac{q^4}{4M^2R^2} \tag{31}$$

being the effective potential that determines the shell motion.

NEWTONIAN LIMITS

To get a good understanding of the meaning of equation (30) in classical mechanics, we can write this equation in the case of Newtonian limits, considering that $\frac{m}{R} \prec\prec 1, \dot{R} \prec\prec 1$ and $\frac{q^2}{R^2} \prec\prec 1$, as

$$m \cong M + \tfrac{1}{2}M\dot{R}^2 - \frac{M\,\mathrm{m}_{-}}{R} - \frac{M^2}{2R} + \frac{M\,q^2}{2R^2} - \frac{q^2}{2R}. \tag{32}$$

This represents, respectively, the sum of the rest energy, kinetic energy, mutual potential energy, gravitational self-energy of the shell, and extra two terms related to the charge q of the shell. The right hand side is not constant but varies according to the quantity *m*. Differentiating equation (29) and using (25-27), then

$$M\ddot{\mathrm{R}} = -\frac{M}{R^2}(m_- + \tfrac{1}{2}M\mathrm{F}_- + \frac{q^2}{2M}\mathrm{F}_+) + \frac{1}{R}(-\dot{M}\mathrm{F}_-\mathrm{F}_+ - \frac{q^2}{RM}\dot{M}\mathrm{F}_+ - \dot{m}_-\mathrm{F}_+ + \dot{m}_+\mathrm{F}_- + \frac{q^2}{RM}\dot{m}_+ + \frac{q}{R}\dot{\mathrm{q}}\mathrm{F}_+). \quad (33)$$

This represents the relativistic version of the second-order equation analysed by Ostriker and Gunn [6] in studying the dynamics of remnants of supernovas by thin spherical shells. The masses m_+ and m_- are functions of υ_+ and υ_- respectively. After taking the Newtonian limit of equation (33) and using

$$\frac{\dot{M}}{\dot{\mathrm{R}}} = -8\pi\,\mathrm{R}\,\mathrm{P} \quad \text{and} \quad \frac{\dot{m}_\pm}{\dot{R}} = -4\pi\,\mathrm{R}^2\,\sigma_\pm,$$

the final expression is

$$M\ddot{\mathrm{R}} \cong -\frac{M}{R^2}(m_- + \tfrac{1}{2}M) + 4\pi R^2(\sigma_- - \sigma_+) + 8\pi R\,\mathrm{p} - \frac{q^2}{\mathrm{M}R}(\frac{M}{2R} + 4\pi R^2\sigma_+ - 8\pi R\,\mathrm{p}) + \frac{q\dot{q}}{R\dot{R}}, \quad (34)$$

where σ_- and σ_+ are the energy density of the radiation at each side of the shell and p is the pressure of the shell. Therefore, all terms described in Ostriker's equation (1) are recovered and have an extra term related to the pressure of the shell and other extra terms related to the charge of the shell.

CONCLUSIONS

The dynamic equation of motion of a radiating Reissner-Nordstrom thin shell is constructed. The final expression of equation (34) with $\dot{q} \to 0$ after taking the Newtonian limit becomes

$$M\ddot{R} \cong -\frac{M}{R^2}(m_- + \tfrac{1}{2}M) + 4\pi\mathrm{R}^2(\sigma_- - \sigma_+) + 8\pi R\,\mathrm{p} - \tfrac{4\pi}{3}R^3\frac{\dot{\sigma}_+}{\dot{R}} + \frac{\mathrm{q}^2}{MR}(8\pi R\,\mathrm{p} - \frac{M}{2R} - 4\pi R^2\sigma_+ - \tfrac{4\pi}{3}R^3\frac{\dot{\sigma}_+}{\dot{R}}).$$

This equation covers all terms in Ostriker's equation with two extra terms related to the pressure of the thin shell and the effect of interstellar medium outside the thin shell ($\dot{\sigma}_+ \equiv \dot{\sigma}_{is}$), and with other extra terms related to the charge of the thin shell.

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