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Sylvester-Padovan-Jacobsthal-type sequences

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Abstract: We define Sylvester-Padovan-Jacobsthal-type sequences of the first and second kinds via the Sylvester matrices which are obtained from the characteristic polynomials of Padovan and Jacobsthal sequences, and then we give miscellaneous properties of these sequences. Also, we obtain cyclic groups and semigroups from the multiplicative order of the generator matrices of the Sylvester-Padovan-Jacobsthal-type sequences of the first-kind and second-kind modulo m. Then we derive relationships between the orders of the cyclic groups obtained and the periods of these sequences modulo m.

Keywords: Sylvester matrix, Padovan sequence, Jacobsthal sequence

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INTRODUCTION

Number-theoretic properties such as those obtained from homogeneous linear recurrence relations relevant to this article have been studied by many authors [1-6]. The study of recurrence sequences in groups began with the earlier work of Wall [7], who investigated the ordinary Fibonacci sequences in cyclic groups. The concept involved was extended to some special linear recurrence sequences by some authors [8-12]. Recently, several authors have obtained the cyclic groups via some special matrices [1, 12-16]. In the present article we define Sylvester-Padovan-Jacobsthal-type sequences of the first and second kinds via Sylvester matrices which are obtained from the characteristic polynomials of Padovan and Jacobsthal sequences. Then we derive the generating functions, the generating matrices, the Binet formulas, the perrmanental and determinantal representations, and the sums of these sequences. Also, we study the Sylvester-Padovan-Jacobsthal-type sequences of the first-kind and second-kind modulo m. Furthermore, we consider the cyclic groups and the semigroups which are generated by the generating matrices of the recurrence sequences defined, and then we obtain relationships between the orders of the cyclic groups obtained and the periods of these sequences modulo m.

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METHODS

Let f and g be polynomials of degree $(k)^{th}$ and $(m)^{th}$ respectively and let these polynomials be given by

$$f = a_k x^k + a_{k-1} x^{k-1} + \cdots + a_1 x + a_0$$

and

$$g = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0.$$

The Sylvester matrix $S_{f,g} = \lfloor S_{ij} \rfloor_{(k+m)\times(k+m)}$ associated with the polynomials f and g is defined as:

$$\begin{bmatrix} a_k & a_{k-1} & \cdots & a_2 & a_1 & a_0 & 0 & 0 & \cdots & 0 \\ 0 & a_k & a_{k-1} & \cdots & a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & 0 & a_k & a_{k-1} & \cdots & a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & a_k & a_{k-1} & \cdots & a_2 & a_1 & a_0 \\ b_m & b_{m-1} & \cdots & b_2 & b_1 & b_0 & 0 & 0 & \cdots & 0 \\ 0 & b_m & b_{m-1} & \cdots & b_2 & b_1 & b_0 & 0 & \cdots & 0 \\ 0 & 0 & b_m & b_{m-1} & \cdots & b_2 & b_1 & b_0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & b_m & b_{m-1} & \cdots & b_2 & b_1 & b_0 \end{bmatrix}.$$

The Padovan sequence is the sequence of integers defined by the initial values $p(0) = p(1) = p(2) = 1$ and recurrence relation

$$P(n) = P(n-2) + P(n-3).$$

It is easy to see that the characteristic polynomial of the Padovan sequence is

$$f(x) = x^3 - x - 1.$$

It is known that the Jacobsthal sequence $\{J_n\}$ is defined recursively by the equation

$$J_n = J_{n-1} + 2J_{n-2}$$

for $n \geq 0$, where $J_0 = 0$ and $J_1 = 1$. The characteristic polynomial of the Jacobsthal sequence is

$$f(x) = x^2 - x - 2.$$

Suppose that the $(n+k)^{\text{th}}$ term of a sequence is defined recursively by a linear combination of the preceding k terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + \cdots + c_{k-1} a_{n+k-1}$$

where $c_0, c_1, \ldots, c_{k-1}$ are real constants. Kalman [17] derived a number of closed-form formulas for the generalised sequence by the companion matrix method as follows. Let the matrix A be defined by

$$A=\left[a_{i,j}\right]_{k\times k}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & c_2 & & c_{k-2} & c_{k-1} \end{bmatrix}.$$

Then

$$A^n\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix}=\begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}$$

for $n \geq 0$.

RESULTS AND DISCUSSION

Sylvester-Padovan-Jacobsthal-Type Numbers

The Sylvester matrix $S_{PJ}=\left\lfloor s_{i,j}\right\rfloor_{5\times 5}$ for the characteristic polynomials of the Padovan sequence and the Jacobsthal sequence is

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 1 & -1 & -2 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 1 & -1 & -2 \end{bmatrix}.$$

Now we consider sequences which are defined by using the diagonal elements of the matrix S_{PJ} and are called the Sylvester-Padovan-Jacobsthal-type sequences of the first and second kinds. The Sylvester-Padovan-Jacobsthal-type sequence of the first kind is defined by integer constants $x_1^1=\cdots=x_4^1=0$ and $x_5^1=1$ and the recurrence relation

$$x_{k+5}^1=x_{k+4}^1+x_{k+3}^1-2x_{k+2}^1-2x_{k+1}^1-2x_k^1 \tag{1}$$

for $k \geq 1$. The Sylvester-Padovan-Jacobsthal-type sequence of the second kind is defined by integer constants $x_1^2=\cdots=x_4^2=0$ and $x_5^2=1$ and the recurrence relation

$$x_{k+5}^2=-2x_{k+4}^2-2x_{k+3}^2-2x_{k+2}^2+x_{k+1}^2+x_k^2 \tag{2}$$

for $k \geq 1$.

We obtain that the generating functions of the Sylvester-Padovan-Jacobsthal-type sequences of the first and second kinds are, respectively,

$$g^1\left(x^1\right)=\frac{\left(x^1\right)^4}{1-x^1-\left(x^1\right)^2+2\left(x^1\right)^3+2\left(x^1\right)^4+2\left(x^1\right)^5}$$

and

$$g^2\left(x^2\right)=\frac{\left(x^2\right)^4}{1+2x^2+2\left(x^2\right)^2+2\left(x^2\right)^3-\left(x^2\right)^4-\left(x^2\right)^5}.$$

By (1) and (2), we can write the following companion matrices:

$$M^{(1)} = \left[m_{i,j}^{(1)} \right]_{5\times 5} = \begin{bmatrix} 1 & 1 & -2 & -2 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$M^{(2)} = \left[m_{i,j}^{(2)} \right]_{5\times 5} = \begin{bmatrix} -2 & -2 & -2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The matrices $M^{(1)}$ and $M^{(2)}$ are said to be the Sylvester-Padovan-Jacobsthal-type matrices of the first and second kinds respectively. By induction on n, we derive

$$\left(M^{(1)}\right)^n = \begin{bmatrix} x_{n+5}^1 & -2\left(x_{n+3}^1 + x_{n+2}^1 + x_{n+1}^1\right) + x_{n+4}^1 & -2\left(x_{n+4}^1 + x_{n+3}^1 + x_{n+2}^1\right) & -2\left(x_{n+4}^1 + x_{n+3}^1\right) & -2x_{n+4}^1 \\ x_{n+4}^1 & -2\left(x_{n+2}^1 + x_{n+1}^1 + x_{n}^1\right) + x_{n+3}^1 & -2\left(x_{n+3}^1 + x_{n+2}^1 + x_{n+1}^1\right) & -2\left(x_{n+3}^1 + x_{n+2}^1\right) & -2x_{n+3}^1 \\ x_{n+3}^1 & -2\left(x_{n+1}^1 + x_{n}^1 + x_{n-1}^1\right) + x_{n+2}^1 & -2\left(x_{n+2}^1 + x_{n+1}^1 + x_{n}^1\right) & -2\left(x_{n+2}^1 + x_{n+1}^1\right) & -2x_{n+2}^1 \\ x_{n+2}^1 & -2\left(x_{n}^1 + x_{n-1}^1 + x_{n-2}^1\right) + x_{n+1}^1 & -2\left(x_{n+1}^1 + x_{n}^1 + x_{n-1}^1\right) & -2\left(x_{n+1}^1 + x_{n}^1\right) & -2x_{n+1}^1 \\ x_{n+1}^1 & -2\left(x_{n-1}^1 + x_{n-2}^1 + x_{n-3}^1\right) + x_{n}^1 & -2\left(x_{n}^1 + x_{n-1}^1 + x_{n-2}^1\right) & -2\left(x_{n}^1 + x_{n-1}^1\right) & -2x_{n}^1 \end{bmatrix}$$

and

$$\left(M^{(2)}\right)^n = \begin{bmatrix} x_{n+5}^2 & -2\left(x_{n+4}^2 + x_{n+3}^2\right) + \left(x_{n+2}^2 + x_{n+1}^2\right) & -2x_{n+4}^2 + x_{n+3}^2 + x_{n+2}^2 & x_{n+4}^2 + x_{n+3}^2 & x_{n+4}^2 \\ x_{n+4}^2 & -2\left(x_{n+3}^2 + x_{n+2}^2\right) + \left(x_{n+1}^2 + x_{n}^2\right) & -2x_{n+3}^2 + x_{n+2}^2 + x_{n+1}^2 & x_{n+3}^2 + x_{n+2}^2 & x_{n+3}^2 \\ x_{n+3}^2 & -2\left(x_{n+2}^2 + x_{n+1}^2\right) + \left(x_{n}^2 + x_{n-1}^2\right) & -2x_{n+2}^2 + x_{n+1}^2 + x_{n}^2 & x_{n+2}^2 + x_{n+1}^2 & x_{n+2}^2 \\ x_{n+2}^2 & -2\left(x_{n+1}^2 + x_{n}^2\right) + \left(x_{n-1}^2 + x_{n-2}^2\right) & -2x_{n+1}^2 + x_{n}^2 + x_{n-1}^2 & x_{n+1}^2 + x_{n}^2 & x_{n+1}^2 \\ x_{n+1}^2 & -2\left(x_{n}^2 + x_{n-1}^2\right) + \left(x_{n-2}^2 + x_{n-3}^2\right) & -2x_{n}^2 + x_{n-1}^2 + x_{n-2}^2 & x_{n}^2 + x_{n-1}^2 & x_{n}^2 \end{bmatrix}$$

for $n \geq 4$. Also, we easily derive that $\det\left(M^{(1)}\right)^n = (-2)^n$ and $\det\left(M^{(2)}\right)^n = 1$.

It is clear that each of the eigenvalues of the matrices $M^{(1)}$ and $M^{(2)}$ is distinct. Let $\{\beta_1^1, \beta_2^1, \beta_3^1, \beta_4^1, \beta_5^1\}$ and $\{\beta_1^2, \beta_2^2, \beta_3^2, \beta_4^2, \beta_5^2\}$ be the sets of the eigenvalues of the matrices $M^{(1)}$ and $M^{(2)}$ respectively and let $V^{(\alpha)}$ be 5×5 Vandermonde matrices as follows:

$$V^{(\alpha)}=\begin{bmatrix}\left(\beta_1^\alpha\right)^4 & \left(\beta_2^\alpha\right)^4 & \left(\beta_3^\alpha\right)^4 & \left(\beta_4^\alpha\right)^4 & \left(\beta_5^\alpha\right)^4\\ \left(\beta_1^\alpha\right)^3 & \left(\beta_2^\alpha\right)^3 & \left(\beta_3^\alpha\right)^3 & \left(\beta_4^\alpha\right)^3 & \left(\beta_5^\alpha\right)^3\\ \left(\beta_1^\alpha\right)^2 & \left(\beta_2^\alpha\right)^2 & \left(\beta_3^\alpha\right)^2 & \left(\beta_4^\alpha\right)^2 & \left(\beta_5^\alpha\right)^2\\ \beta_1^\alpha & \beta_2^\alpha & \beta_3^\alpha & \beta_4^\alpha & \beta_5^\alpha\\ 1 & 1 & 1 & 1 & 1\end{bmatrix}$$

where $\alpha=1,2$. Let

$$W^{(\alpha)}(i,j)=\begin{bmatrix}\left(\beta_1^\alpha\right)^{n+5-i}\\ \left(\beta_2^\alpha\right)^{n+5-i}\\ \left(\beta_3^\alpha\right)^{n+5-i}\\ \left(\beta_4^\alpha\right)^{n+5-i}\\ \left(\beta_5^\alpha\right)^{n+5-i}\end{bmatrix}$$

and let $V^{(\alpha)}(i,j)$ be a 5×5 matrix obtained from $V^{(\alpha)}$ by replacing the j^{th} column of $V^{(\alpha)}$ by $W^{(\alpha)}(i,j)$ for $\alpha=1,2$.

Theorem 1. For $n\geq 4$ and $\alpha=1,2$,

$$m_{i,j}^{(\alpha),n}=\frac{\det V^{(\alpha)}(i,j)}{\det V^{(\alpha)}},$$

where $\left(M^{(\alpha)}\right)^n=\left[m_{i,j}^{(\alpha),n}\right]$.

Proof. Since $\{\beta_1^1,\beta_2^1,\beta_3^1,\beta_4^1,\beta_5^1\}$ and $\{\beta_1^2,\beta_2^2,\beta_3^2,\beta_4^2,\beta_5^2\}$ are distinct, the matrices $M^{(1)}$ and $M^{(2)}$ are diagonalisable. Thus, we easily see that $M^{(\alpha)}V^{(\alpha)}=V^{(\alpha)}D^{(\alpha)}$ and $D^{(\alpha)}=\left(\beta_1^\alpha,\beta_2^\alpha,\beta_3^\alpha,\beta_4^\alpha,\beta_5^\alpha\right)$ for $\alpha=1,2$. Since $\det V^{(1)}\neq 0$ and $\det V^{(2)}\neq 0$, the matrices $V^{(1)}$ and $V^{(2)}$ are invertible. Then we obtain $\left(V^{(\alpha)}\right)^{-1}M^{(\alpha)}V^{(\alpha)}=D^{(\alpha)}$ for $\alpha=1,2$. Thus, the matrices $M^{(1)}$ and $M^{(2)}$ are similar to $D^{(\alpha)}$, and $\left(M^{(\alpha)}\right)^n V^{(\alpha)}=V^{\alpha}\left(D^{(\alpha)}\right)^n$ for $n\geq 4$ and $\alpha=1,2$. Then we can write the following linear system of equations:

$$\begin{cases} m_{i,1}^{(\alpha),n}\left(\beta_1^\alpha\right)^4+m_{i,2}^{(\alpha),n}\left(\beta_1^\alpha\right)^3+\cdots+m_{i,5}^{(\alpha),n}=\left(\beta_1^\alpha\right)^{n+5-i}\\ m_{i,1}^{(\alpha),n}\left(\beta_2^\alpha\right)^4+m_{i,2}^{(\alpha),n}\left(\beta_2^\alpha\right)^3+\cdots+m_{i,5}^{(\alpha),n}=\left(\beta_2^\alpha\right)^{n+5-i}\\ \qquad\vdots\\ m_{i,1}^{(\alpha),n}\left(\beta_5^\alpha\right)^4+m_{i,2}^{(\alpha),n}\left(\beta_5^\alpha\right)^3+\cdots+m_{i,5}^{(\alpha),n}=\left(\beta_5^\alpha\right)^{n+5-i}\end{cases}$$

for $n\geq 4$ and $\alpha=1,2$. Therefore, for each $i,j=1,2,\ldots,5$, we obtain

$$m_{i,j}^{(\alpha),n}=\frac{\det V^{(\alpha)}(i,j)}{\det V^{(\alpha)}}.\qquad\square$$

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Then we can give the Binet formulas for the Sylvester-Padovan-Jacobsthal-type sequences of the first and second kinds by the following corollary.

Corollary 1. Let x_n^1 and x_n^2 be the n^{th} Sylvester-Padovan-Jacobsthal-type numbers of the first and second kinds respectively. Then

$$x_n^1 = -\frac{\det V^{(1)}(5,5)}{2\det V^{(1)}}$$

and

$$x_n^2 = \frac{\det V^{(2)}(5,5)}{\det V^{(2)}}.$$

□

Now we consider the permanental representations of the Sylvester-Padovan-Jacobsthal-type sequences of the first and second kinds.

Definition 1. Let $M = \left[m_{i,j}\right]$ be a $u \times v$ real matrix and let $r^1, r^2, \ldots, r^u$ and $c^1, c^2, \ldots, c^v$ be the row and column vectors of M respectively. If r^α contains exactly two non-zero entries, then M is contractible on row α. Similarly, M is contractible on column β provided that c^β contains exactly two non-zero entries. □

Suppose that $x_1, x_2, \ldots, x_u$ are row vectors of the matrix M. If M is contractible in the k^{th} column such that $m_{i,k} \neq 0,\ m_{j,k} \neq 0$ and $i \neq j$, then the $(u-1)\times(v-1)$ matrix $M_{ij:k}$ is obtained from M by replacing the i^{th} row with $m_{i,k}x_j + m_{j,k}x_i$ and deleting the j^{th} row. The k^{th} column is called the contraction in the k^{th} column relative to the i^{th} row and the j^{th} row. Brualdi and Gibson [18] obtained that $per(M) = per(N)$ if M is a real matrix of order $\alpha > 1$ and N is a contraction of M.

Let $S^{(1)}(u) = \left[s_{i,j}^{(1),u}\right]$ be the $u \times u$ super-diagonal matrix with $s_{i,i}^{(1),u} = s_{i,i+1}^{(1),u} = s_{i+1,i}^{(1),u} = 1$, $s_{i,i+2}^{(1),u} = s_{i,i+3}^{(1),u} = s_{i,i+4}^{(1),u} = -2$, and let $S^{(2)}(u) = \left[s_{i,j}^{(2),u}\right]$ be the $u \times u$ super-diagonal matrix with $s_{i,i+3}^{(2),u} = s_{i,i+4}^{(2),u} = s_{i+1,i}^{(2),u} = 1$, $s_{i,i}^{(2),u} = s_{i,i+1}^{(2),u} = s_{i,i+2}^{(2),u} = -2$ for all i, j and 0 otherwise. Clearly,

$$S^{(1)}(u) = \begin{bmatrix} 1 & 1 & -2 & -2 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 1 & -2 & -2 & -2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & 1 & -2 & -2 & -2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 1 & 1 & -2 & -2 & -2 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & 1 & -2 & -2 & -2 \\ \vdots & & \ddots & & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$

and

$$S^{(2)}(u)=\begin{bmatrix} -2 & -2 & -2 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & -2 & -2 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & -2 & -2 & 1 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -2 & -2 & -2 & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -2 & -2 & -2 & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & -2 & -2 \\ \vdots & \vdots & \vdots & & \ddots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}.$$

Then we have the following Theorem.

Theorem 2. (i) For $u \geq 1$,

$$perS^{(1)}(u) = x^{1}_{u+5}$$

where $perS^{(1)}(1) = 1$.

(ii) For $u \geq 1$,

$$perS^{(2)}(u) = x^{2}_{u+5}$$

where $perS^{(2)}(1) = -2$.

Proof. (i) First we start with considering the case $u < 5$. The matrices $S^{(1)}(2), S^{(1)}(3)$ and $S^{(1)}(4)$ are reduced to the following forms:

$$S^{(1)}(2)=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$S^{(1)}(3)=\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and

$$S^{(1)}(4)=\begin{bmatrix} 1 & 1 & -2 & -2 \\ 1 & 1 & 1 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

It is easy to see that $perS^{(1)}(2) = 2$, $perS^{(1)}(3) = 1$ and $perS^{(1)}(4) = -1$. From definition of the Sylvester-Padovan-Jacobsthal-type number of the first kind, it is clear that $x^{1}_{7} = 2$, $x^{1}_{8} = 1$ and $x^{1}_{9} = -1$. So we have the conclusion for $u < 5$. Let the equation hold for $u \geq 5$; then we show that the equation holds for $u+1$. If we expand $perS^{(1)}(u)$ by the Laplace expansion of permanent according to the first row, then we obtain

$$perS^{(1)}(u+1) = perS^{(1)}(u) + perS^{(1)}(u-1) - 2perS^{(1)}(u-2) - 2perS^{(1)}(u-3) - 2perS^{(1)}(u-4).$$

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Since $perS^{(1)}(u) = x^1_{u+5}$, $perS^{(1)}(u-1) = x^1_{u+4}$, $perS^{(1)}(u-2) = x^1_{u+3}$, $perS^{(1)}(u-3) = x^1_{u+2}$ and $perS^{(1)}(u-4) = x^1_{u+1}$, we easily obtain that $perS^{(1)}(u+1) = x^1_{u+6}$. So the proof is complete.

(ii) The proof is similar to the above and is omitted. □

Let $u > 5$ and let $P^{(1)}(u) = \left[p^{(1),u}_{i,j} \right]$ and $P^{(2)}(u) = \left[p^{(2),u}_{i,j} \right]$ be the $u \times u$ matrices, defined respectively by

$$p^{(1),u}_{i,j} = \begin{cases} 1 & \begin{array}{c} \text{if } \ i = k \text{ and } j = k \ \text{ for } 1 \le k \le u, \\ i = k \text{ and } j = k+1 \quad \text{for } 1 \le k \le u-2 \\ \text{and} \\ i = k+1 \text{ and } j = k \quad \text{for } 1 \le k \le u-3, \end{array} \\ \\ -2 & \begin{array}{c} \text{if } \ i = k \text{ and } j = k+2 \ \text{ for } 1 \le k \le u-4, \\ i = k \text{ and } j = k+3 \quad \text{for } 1 \le k \le u-4 \\ \text{and} \\ i = k \text{ and } j = k+4 \quad \text{for } 1 \le k \le u-4, \end{array} \\ \\ 0 & \qquad \text{otherwise.} \end{cases}$$

and

$$p^{(2),u}_{i,j} = \begin{cases} 1 & \begin{array}{c} \text{if } i = k \text{ and } j = k+3 \quad \text{for } 1 \le k \le u-3, \\ i = k \text{ and } j = k+4 \quad \text{for } 1 \le k \le u-4, \\ i = k+1 \text{ and } j = k \quad \text{for } 1 \le k \le u-3, \\ i = u-1 \text{ and } j = u-1, \ j = u \\ \text{and} \\ i = u \text{ and } j = u, \end{array} \\ \\ -2 & \begin{array}{c} \text{if } i = k \text{ and } j = k \quad \text{for } 1 \le k \le u-2, \\ i = k \text{ and } j = k+1 \quad \text{for } 1 \le k \le u-2 \\ \text{and} \\ i = k \text{ and } j = k+2 \quad \text{for } 1 \le k \le u-2, \end{array} \\ \\ 0 & \qquad \text{otherwise.} \end{cases}$$

Assume that the $u \times u$ matrices $Z^{(\alpha)}(u) = \left[z^{(\alpha),u}_{i,j} \right]$ are defined by

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$$Z^{(\alpha)}(u)=\begin{array}{c} \begin{array}{ccccc} & & (u-2)\text{th} & & \\ & & \downarrow & & \end{array} \\ \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & & & & & \\ 0 & & & P^{(\alpha)}(u-1) & & \\ \vdots & & & & & \\ 0 & & & & & \end{bmatrix} \end{array}$$

where $u>6$ and $\alpha=1,2$. Then we can give more general results by using permanental representations other than the above.

Theorem 3. **(i)** For $u>5$ and $\alpha=1,2$,

$$perP^{(\alpha)}(u)=x_{u+3}^{\alpha}.$$

(ii) For $u>6$ and $\alpha=1,2$,

$$perZ^{(\alpha)}(u)=\sum_{k=1}^{u+2}x_k^{\alpha}.$$

Proof. **(i)** Let us consider the matrix $P^{(2)}(u)$ and let the equation hold for $u>5$. Then we show that the equation holds for $u+1$. If we expand $perP^{(2)}$ by the Laplace expansion of permanent according to the first row, then we obtain

$$perP^{(2)}(u+1)=-2perP^{(2)}(u)-2perP^{(2)}(u-1)-2perP^{(2)}(u-2)+perP^{(2)}(u-3)+perP^{(2)}(u-4)$$

Also, since

$$perP^{(2)}(u)=x_{u+3}^2,\ perP^{(2)}(u-1)=x_{u+2}^2,\ perP^{(2)}(u-2)=x_{u+1}^2,\ perP^{(2)}(u-3)=x_u^2,\ perP^{(2)}(u-4)=x_{u-1}^2,$$

it is clear that

$$perP^{(2)}(u+1)=x_{u+4}^2.$$

So we have the conclusion. The proof for the matrix $P^{(1)}(u)$ is similar.

(ii) If we extend $perZ^{(\alpha)}(u)$ with respect to the first row, we write

$$perZ^{(\alpha)}(u)=perZ^{(\alpha)}(u-1)+perP^{(\alpha)}(u-1)$$

for $\alpha=1,2$. By induction on u, taking into consideration the results of Theorem 2 and part (i) in Theorem 3, the conclusion is easily seen. □

Let the notation $M\circ K$ denote the Hadamard product of M and K. A matrix M is called convertible if there is a $u\times u$ (1, -1)-matrix K such that $perM=\det(M\circ K)$. Let $u>5$ and let T be the $u\times u$ matrix defined by

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$$T=\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & -1 & 1 \end{bmatrix}.$$

It is easy to see that $perS^{(\alpha)}(u)=\det\left(S^{(\alpha)}(u)\circ T\right)$, $perP^{(\alpha)}(u)=\det\left(P^{(\alpha)}(u)\circ T\right)$ and $perZ^{(\alpha)}(u)=\det\left(Z^{(\alpha)}(u)\circ T\right)$ for $u>5$ and $\alpha=1,2$. Then we have the following useful results.

Corollary 2. For $u>5$ and $\alpha=1,2$,

$$\det\left(S^{(\alpha)}(u)\circ T\right)=x_{u+5}^{\alpha},$$
$$\det\left(P^{(\alpha)}(u)\circ T\right)=x_{u+3}^{\alpha}$$

and

$$\det\left(Z^{(\alpha)}(u)\circ T\right)=\sum_{k=1}^{u+2}x_k^{\alpha}. \qquad \square$$

Now we consider the sums of the Sylvester-Padovan-Jacobsthal-type sequences of the first and second kinds. Let

$$G_n=\sum_{k=1}^{n}x_k^{\alpha}$$

for $n\geq 1$ and $\alpha=1,2$. Suppose that $A_6^{(\alpha)}$ and $A_6^{(\alpha),n}$ are the 6×6 matrices such that

$$A_6^{(\alpha)}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & & & & & \\ 0 & & & & & \\ 0 & & & M^{(\alpha)} & & \\ 0 & & & & & \\ 0 & & & & & \end{bmatrix}$$

and

$$A_6^{(\alpha),n}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ G_{n+4} & & & & & \\ G_{n+3} & & & & & \\ G_{n+2} & & & \left(M^{(\alpha)}\right)^n & & \\ G_{n+1} & & & & & \\ G_n & & & & & \end{bmatrix}.$$

Then it can be shown by induction that $\left(A_6^{(\alpha)}\right)^n=A_6^{(\alpha),n}$.

Cyclic Groups and Semigroups via Matrices $M^{(1)}$ and $M^{(2)}$

For a given matrix $D = \lfloor d_{ij} \rfloor$ with d_{ij}'s being integers, $D(modm)$ means that each element of D is a reduced modulo m, i.e. $D(modm) = \left(d_{ij}\,(modm)\right)$. Let us consider the set $\langle D \rangle_m = \{D^i(modm) | i \geq 0\}$. If $\gcd(m, \det D) = 1$, then $\langle D \rangle_m$ is a cyclic group; if $\gcd(m, \det D) \neq 1$, then the set $\langle D \rangle_m$ is a semigroup. Since $\det M^{(1)} = -2$, it is clear that the set $\left\langle M^{(1)} \right\rangle_m$ is a cyclic group when m is a positive odd integer; otherwise $\left\langle M^{(1)} \right\rangle_m$ is a semigroup. Since $\det M^{(2)} = 1$, it is clear that the set $\left\langle M^{(2)} \right\rangle_m$ is a cyclic group for every positive integer m.

Theorem 4. Let r be a prime and let $\langle G \rangle_{r^t}$ be any of the cyclic groups $\left\langle M^{(1)} \right\rangle_{r^t}$ and $\left\langle M^{(2)} \right\rangle_{r^t}$. If u is the largest positive integer such that $\left|\langle G \rangle_r\right| = \left|\langle G \rangle_{r^u}\right|$, then $\left|\langle G \rangle_{r^v}\right| = r^{v-u} \cdot \left|\langle G \rangle_r\right|$. In particular, if $\left|\langle G \rangle_r\right| \neq \left|\langle G \rangle_{r^2}\right|$, then $\left|\langle G \rangle_{r^v}\right| = r^{v-1} \cdot \left|\langle G \rangle_r\right|$.

Proof. Let us consider the cyclic group $\left\langle M^{(1)} \right\rangle_{r^t}$. Then $\gcd(r, -2) = 1$; that is, r is an odd prime. Suppose that b is a positive integer and $\left|\left\langle M^{(1)} \right\rangle_{r^t}\right|$ is denoted by $P(r^t)$. If $\left(M^{(1)}\right)^{P(r^{b+1})} \equiv I \left(\bmod r^{b+1}\right)$, then $\left(M^{(1)}\right)^{P(r^{b+1})} \equiv I \left(\bmod r^{b}\right)$, where I is a 5×5 identity matrix. Thus, we obtain that $P(r^b)$ divides $P(r^{b+1})$. On the other hand, writing $\left(M^{(1)}\right)^{P(r^b)} = I + \left(m_{ij}^{(b)} \cdot r^b\right)$, by the binomial theorem we have

$$\left(M^{(1)}\right)^{P(r^b)\cdot r} = \left(I + \left(m_{ij}^{(b)} \cdot r^b\right)\right)^r = \sum_{i=0}^{r} \binom{r}{i} \left(m_{ij}^{(b)} \cdot r^b\right)^i \equiv I \left(\bmod r^{b+1}\right).$$

So we have that $P(r^{b+1})$ divides $P(r^b) \cdot r$. Thus, $P(r^{b+1}) = P(r^b)$ or $P(r^{b+1}) = P(r^b) \cdot r$. It is clear that $P(r^{b+1}) = P(r^b) \cdot r$ holds if and only if there exists an integer $m_{ij}^{(b)}$ which is not divisible by r. Since u is the largest positive integer such that $P(r) = P(r^u)$, we have $P(r^u) \neq P(r^{u+1})$. Then, there exists an integer $m_{ij}^{(u+1)}$ which is not divisible by r. So we have that $P(r^{u+1}) \neq P(r^{u+2})$. To complete the proof we may use an inductive method on u . □

It is well known that a sequence is *periodic* if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the shortest repeating subsequence is called the period of the sequence. In particular, if the first k elements in the sequence form a repeating subsequence, then the sequence is *simply periodic* and its period is k.

Reducing the Sylvester-Padovan-Jacobsthal-type sequences of the first and second kinds by a modulus m, we get the repeating sequences respectively denoted by

$$\left\{x_k^{1,m}\right\} = \left\{x_1^{1,m}, x_2^{1,m}, \ldots, x_i^{1,m}, \ldots\right\}$$

and

$$\left\{x_k^{2,m}\right\} = \left\{x_1^{2,m}, x_2^{2,m}, \ldots, x_i^{2,m}, \ldots\right\},$$

where $x_i^{1,m} = x_i^1\,(modm)$ and $x_i^{2,m} = x_i^2\,(modm)$. They have the same recurrence relations as in (1) and (2) respectively.

Theorem 5. For $\alpha = 1,2$, the sequences $\{x_k^{\alpha,m}\}$ are periodic. In particular, the sequence $\{x_k^{2,m}\}$ is simply periodic.

Proof. Let $Q = \{(q_1, q_2, q_3, q_4, q_5) \mid q_i$' s, being integers such that $0 \le q_i \le m-1\}$. Since there are m^5 distinct 5-tuples of elements of Z_m, at least one of the 5-tuples appears twice in the sequences $\{x_k^{\alpha,m}\}$. Thus, the subsequence following this 5-tuple repeats; hence the sequences $\{x_k^{1,m}\}$ and $\{x_k^{2,m}\}$ are periodic. Let

$$x_{i+1}^{2,m} \equiv x_{j+1}^{2,m},\ x_{i+2}^{2,m} \equiv x_{j+2}^{2,m},\ x_{i+3}^{2,m} \equiv x_{j+3}^{2,m},\ x_{i+4}^{2,m} \equiv x_{j+4}^{2,m},\ x_{i+5}^{2,m} \equiv x_{j+5}^{2,m}$$

such that $i > j$; then $i \equiv j(mod5)$. From the definition, we can easily obtain

$$x_i^{2,m} \equiv x_j^{2,m}, x_{i-1}^{2,m} \equiv x_{j-1}^{2,m}, \ldots, x_{i-j+1}^{2,m} \equiv x_1^{2,m},$$

which implies that the sequence $\{x_k^{2,m}\}$ is simply periodic. □

We next denote the period of the sequences $\{x_k^{\alpha,m}\}$ by $Px_k^{\alpha,m}$.

Theorem 6. If m has the prime factorisation $m = \prod_{i=1}^{u} (p_i)^{r_i}$, $(u \ge 1)$, then $Px_k^{\alpha,m}$ is equal to the least common multiple of $Px_k^{\alpha,(p_i)^{r_i}}$.

Proof. It is clear that the sequences $\left\{x_k^{\alpha,(p_i)^{r_i}}\right\}$ repeats only after blocks of length $l \cdot Px_k^{\alpha,(p_i)^{r_i}}$, where l is a natural number. Since $Px_k^{\alpha,m}$ is a period of the sequences $\{x_k^{\alpha,m}\}$, the sequences $\left\{x_k^{\alpha,(p_i)^{r_i}}\right\}$ repeats after $Px_k^{\alpha,m}$ terms for all values i. Therefore, $Px_k^{\alpha,m}$ is of the form $l \cdot Px_k^{\alpha,(p_i)^{r_i}}$ for all values of i, and since any such number gives a period of $Px_k^{\alpha,m}$, we can conclude that

$$Px_k^{\alpha,m} = lcm\left[Px_k^{\alpha,(p_1)^{r_1}}, \ldots, Px_k^{\alpha,(p_u)^{r_u}}\right].$$ □

CONCLUSIONS

We have defined Sylvester-Padovan-Jacobsthal-type sequences of the first and second kinds. Firstly, we gave the generating functions, the generating matrices, the Binet formulas, the perrmanental and determinantal representations, and the sums of the Sylvester-Padovan-Jacobsthal-type numbers of the first and second kinds. Furthermore, we studied the Sylvester-Padovan-Jacobsthal-type sequences of the first-kind and second-kind modulo m and obtained the cyclic groups and semigroups from generating matrices of the sequences of the first and second kinds. Finally, we produced relationships between the orders of the cyclic groups obtained and the periods of these sequences modulo m.

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