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Full Paper

Fourth-order method for singularly perturbed singular boundary value problems using non-polynomial spline

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Abstract: This paper envisages a fourth-order finite difference method with reference to the solution of a class of singularly perturbed singular boundary value problems especially on a uniform mesh. The non-polynomial spline forms the tool for the solution of the problems. The discretisation equation of the problems are developed using the condition of continuity for the first-order derivatives of the non- polynomial spline at the interior nodes and it is not valid at the singularity. Hence at the singularity, the boundary value problem is modified in order to get a three-term relation. The tridiagonal scheme of the method is processed using discrete invariant imbedding algorithm. The convergence of the method is analysed and maximum absolute errors in the solution are tabulated. Root mean square errors in the solution of the examples are presented in comparison to the methods chosen from the literature to establish the proposed method.

Keywords: singularly perturbed two-point singular boundary value problem, interior nodes, singular point, non-polynomial spline, boundary layer

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INTRODUCTION

We consider a class of singularly perturbed two-point singular boundary value problems of the form:

$$\varepsilon y''(x) = \frac{k}{x} y'(x) + q(x)y(x) + r(x), \qquad 0 \le x \le 1, \tag{1}$$

with boundary conditions $\quad y(0) = \gamma_1 \quad$ and $\quad y(1) = \gamma_2$, (2)

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where $0 < \varepsilon << 1$, $q(x)$ and $r(x)$ are bounded continuous functions in (0, 1), and γ_1, γ_2 are finite constants. Let $p(x) = \frac{k}{x}$. If $p(x) \geq \overline{M} > 0$ throughout the domain [0, 1], where $\overline{M}$ is a positive constant, then the boundary layer exists in the neighbourhood of $x = 0$. If $p(x) \leq \overline{N} < 0$ throughout the interval [0, 1], where $\overline{N}$ is a negative constant, then the boundary layer will be in the neighbourhood of $x = 1$.

This class of problems frequently occurs in many areas of applied mathematics such as fluid mechanics, elasticity, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction diffusion process, geophysics and many other areas. Equations of this type exhibit solutions with layers; that is, the domain of solution of the problem contains narrow regions where the solution derivatives are extremely large. The numerical treatment of these problems gives major computational difficulties due to the presence of boundary and/or interior layers. A wide variety of books have been published, describing various methods for solving singularly perturbed two-point boundary value problems. Among these, we mention Henriei [1], O'Malley [2], Bender and Orszag [3] and Kreiss and Kreiss [4]. Bava [5] investigated a fourth-order difference scheme via cubic spline in compression for the solution of singular perturbation problems. Kadalbajoo and Aggarwal [6] proposed a fitted mesh B-spline method for singular singularly perturbed boundary value problems. Kadalbajoo and Patidar [7] derived some difference schemes for singularly perturbed problems using spline in compression. Kadalbajoo and Reddy [8] have discussed a numerical method via deviating arguments to solve linear singular perturbation problems. Mohanty et al. [9, 10] and Mohanty and Aurora [11] have established various methods based on tension spline and compression spline methods, both on a uniform and non-uniform mesh, for singularly perturbed two-point singular boundary value problems. Rashidinia and Ghasemi [12] used cubic spline solution of singularly perturbed two-point boundary value problems on a uniform mesh.

The approach presented in this paper has the advantage over finite difference methods in that it provides continuous approximations not only for $y(x)$ but also for y', y'' and higher derivatives at every point of the range of integration. Also, the C^∞- differentiability of the trigonometric part of non-polynomial splines compensates for the loss of smoothness inherited by polynomial splines. Besides, a new parameter ω is introduced in this method to achieve the desired fourth-order convergence for the problems represented by Eq. (1).

NON-POLYNOMIAL SPLINE METHOD

The domain of the integration [*a, b*] is decomposed into *N* equal subintervals with mesh size $h = \frac{1}{N}$, so that $x_i = a + ih,\ i = 0, 1, \ldots, N$ are the nodes with $a = x_0, b = x_N$. Let $y(x)$ be the exact solution and y_i be an approximation to $y(x_i)$ by the non-polynomial cubic spline $S_i(x)$ passing through the points (x_i, y_i) and (x_{i+1}, y_{i+1}). Here $S_i(x)$ satisfies interpolatory conditions at x_i and x_{i+1}; also, the continuity of first derivative at the common nodes (x_i, y_i) is fulfilled. For each i^{th} subinterval, the cubic non-polynomial spline function $S_i(x)$ has the form:

$$S_i(x) = a_i + b_i(x - x_i) + c_i \sin \tau(x - x_i) + d_i \cos \tau(x - x_i),\quad i = 0,\ 1, \ldots, N-1, \tag{3}$$

where a_i, b_i, c_i and d_i are constants and τ is a free parameter.

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A non-polynomial function $S_i(x)$ of class $C^2[a,b]$ interpolating *y*(*x*) at the grid points x_i, *i* = 0, 1,...,*N* depends on a parameter τ and reduces to ordinary cubic spline in [*a, b*] as $\tau \to 0$. To derive an expression for the coefficients of Eq. (3) in terms of y_i, y_{i+1}, M_i and M_{i+1}, the following are defined:

$$S_i(x_i) = y_i, S_i(x_{i+1}) = y_{i+1},$$
$$S''(x_i) = M_i, S''(x_{i+1}) = M_{i+1}.$$

Using algebraic manipulation, the following expressions are obtained for the coefficients:

$$a_i = y_i + \frac{M_i}{\tau^2},\ b_i = \frac{y_{i+1} - y_i}{h} + \frac{M_{i+1} - M_i}{\tau\theta},$$
$$c_i = \frac{M_i \cos\theta - M_{i+1}}{\tau^2 \sin\theta},\ d_i = -\frac{M_i}{\tau^2}$$

where $\theta = \tau h$, for *i* = 0, 1, ... , *N*-1. Using the continuity of the first derivative at (x_i, y_i), that is $S'_{i-1}(x_i) = S'_i(x_i)$, we get the following relations for *i* =1,2,...,*N*-1 :

$$\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \tag{4}$$

where

$$\alpha = \frac{-1}{\theta^2} + \frac{1}{\theta\sin\theta},\ \beta = \frac{1}{\theta^2} - \frac{\cos\theta}{\theta\sin\theta},\ M_j = y''(x_j),\ j = i-1, i, i+1 \text{ and } \theta = \tau h.$$

NUMERICAL METHOD

At the grid points x_i, Eq. (1) may be discretised by

$$\varepsilon y''_i = p(x_i)y'_i + q(x_i)y_i + r_i .$$

Using spline's second derivatives, we have

$$\varepsilon M_j = p(x_j)y'_j(x) + q(x_j)y(x_j) + r(x_j) \text{ for } j = i-1,\ i,\ i+1 . \tag{5}$$

Using Eq.(5) in Eq.(4) and the following approximations for the first derivative of y [5]:

$$y'_{i+1} \approx \frac{y_{i-1} - 4y_i + 3y_{i+1}}{2h},$$
$$y'_{i-1} \approx \frac{-3y_{i-1} + 4y_i - y_{i+1}}{2h},$$
$$y'_i \cong \left(\frac{1 + 2\omega h^2 q_{i+1} + \omega h[3p_{i+1} + p_{i-1}]}{2h}\right) y_{i+1} - 2\omega\ [p_{i+1} + p_{i-1}] y_i \tag{6}$$
$$-\left(\frac{1 + 2\omega h^2 q_{i-1} - \omega h[p_{i+1} + 3p_{i-1}]}{2h}\right) y_{i-1} + \omega\, h[r_{i+1} - r_{i-1}] ,$$

we get the tridiagonal system:

$$E_{i-1}y_{i-1} + F_i\ y_i + G_{i+1}y_{i+1} = H_i \quad \text{for } i = 1,2,..., N-1 \tag{7}$$

where

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$$E_{i-1} = -\varepsilon - \frac{3}{2}\alpha\, p_{i-1}\, h + \beta\, p_i h^2 \omega\left[p_{i+1} + 3p_{i-1}\right] - 2\omega\, p_i \beta\, h^3 q_{i-1} + \frac{\alpha}{2} p_{i+1} h + \alpha\, q_{i-1} h^2 - h\,\beta\, p_i \;,$$

$$F_i = 2\varepsilon + 2\alpha\, p_{i-1}\, h - 4\beta\, p_i h^2 \omega\left[p_{i+1} + p_{i-1}\right] - 2\alpha\, p_{i+1} h + 2\beta\, q_i h^2 ,$$

$$G_{i+1} = -\varepsilon - \frac{\alpha}{2}\, p_{i-1}\, h + \beta\, p_i h^2 \omega\left[3p_{i+1} + p_{i-1}\right] + 2\omega h^3 \beta\, p_i q_{i+1} + \frac{3}{2}\alpha\, p_{i+1} h + \alpha\, q_{i+1} h^2 + h\beta\, p_i \;,$$

$$H_i = -h^2\left[\left(\alpha - 2\omega\beta\, p_i h\right) r_{i-1} + 2\beta\, r_i + \left(\alpha + 2\omega\beta\, p_i h\right) r_{i+1}\right] \qquad ,$$

$$p(x_i) = p_i \;, q(x_i) = q_i \;, r(x_i) = r_i \text{ for } i = 0, 1, \ldots, N \;.$$

For i = 1, the coefficients y_{i-1}, y_i and y_{i+1} in Eq. (7) are not defined; thus, we need to develop a formula for this case. Using L-Hospital rule and Eq. (4), we get the following three-term formula for i = 1:

$$\left[-1 + \frac{\alpha\, h^2 q_0}{\varepsilon - k}\right] y_0 + \left[2 + \frac{2\beta\, h^2 q_1}{\varepsilon - k}\right] y_1 + \left[-1 + \frac{\alpha\, h^2 q_2}{\varepsilon - k}\right] y_2 = -\frac{h^2}{\varepsilon - k}\left[\alpha\, r_0 + 2\beta\, r_1 + \alpha\, r_2\right]. \qquad (8)$$

Using discrete invariant imbedding algorithm [8], the tridiagonal system Eq.(7) together with Eq.(8) for i = 1,2,…,N-1 is solved in order to get the approximations $y_{1,} y_2, \ldots, y_{N-1}$ of the solution $y(x)$ at $x_1, x_2, \ldots, x_{N-1}$.

TRUNCATION ERROR

The local truncation error associated with the scheme developed in Eq. (7) is

$$T_i(h) = \left[-1 + 2(\alpha + \beta)\right]\varepsilon h^2 y''(x_i) + \left\{\left[\left(4\omega\varepsilon + \frac{1}{3}\right)\beta - \frac{2\alpha}{3}\right] p(x_i) y'''(x_i) + \left(-1 + 12\alpha\right)\frac{\varepsilon}{12} y^{(4)}(x_i)\right\} h^4 + O(h^6).$$

Thus, for different values of α , β, ω in the scheme of Eq. (7), the following different orders are indicated:

(i) For any choice of arbitrary α and β with $\alpha + \beta = \frac{1}{2}$ and for any value of ω, the scheme of Eq. (7) gives the second-order method;

(ii) For $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$ and $\omega = -\frac{1}{20\varepsilon}$, from Eq. (7) the fourth-order method is derived.

CONVERGENCE ANALYSIS

Incorporating the boundary conditions (Eq. 2), the system of Eq. (7) and (8) can be written in the matrix form as:

$$\left(D + P\right)Y + Q + T(h) = 0 \;, \qquad (9)$$

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where $$D=[-\varepsilon,\ 2\varepsilon,\ -\varepsilon]=\begin{bmatrix} 2 & -1 & 0 & 0 & \ldots\ldots & 0 \\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & \ldots\ldots & 0 \\ 0 & -\varepsilon & 2\varepsilon & -\varepsilon & \ldots\ldots & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & -\varepsilon & 2\varepsilon \end{bmatrix}$$

and

$$P=[z_i, v_i, w_i]=\begin{bmatrix} v_1^* & w_1^* & 0 & 0 & \ldots\ldots & 0 \\ z_2 & v_2 & w_2 & 0 & \ldots\ldots & 0 \\ 0 & z_3 & v_3 & w_3 & \ldots\ldots & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & z_{N-1} & v_{N-1} \end{bmatrix},$$

in which $v_1^* = 2+\dfrac{2\beta h^2 q_1}{\varepsilon-k}$, $w_1^* = -1+\dfrac{\alpha h^2 q_2}{\varepsilon-k}$,

$z_i = -\dfrac{3}{2}\alpha\, p_{i-1}\, h+\beta\, p_i h^2\omega[p_{i+1}+3p_{i-1}]-2\omega\, p_i\beta\, h^3 q_{i-1}+\dfrac{\alpha}{2}\, p_{i+1}h+\alpha\, q_{i-1}h^2 \text{ - } h\beta\, p_i$,

$v_i = 2\alpha\, p_{i-1}\, h-4\beta\, p_i h^2\omega[p_{i+1}+p_{i-1}]-2\alpha\, p_{i+1}h+2\beta\, q_i h^2$,

$w_i = -\dfrac{\alpha}{2}\, p_{i-1}\, h+\beta\, p_i h^2\omega[3p_{i+1}+p_{i-1}]+2\omega h^3\beta\, p_i q_{i+1}+\dfrac{3}{2}\alpha\, p_{i+1}h+\alpha\, q_{i+1}h^2+h\beta\, p_i$, for $i=2,3,\ldots,N-1$

and

$$Q=\left[\frac{h^2}{\varepsilon-k}(\alpha r_0+2\beta r_1+\alpha r_2)+\left(-1+\frac{\alpha h^2 q_0}{\varepsilon-k}\right)\gamma_0, q_2, q_3,\ldots,q_{N-1}+w_{N-1}\gamma_1\right],$$

wherein $q_i = h^2\left[(\alpha-2\omega\beta\, p_i h)r_{i-1}+2\beta\, r_i+(\alpha+2\omega\beta\, p_i h)r_{i+1}\right]$ for $i=2,3,\ldots,N-1$, $T(h)=0(h^6)$ for $\alpha=\dfrac{1}{12}, \beta=\dfrac{5}{12}$, $\omega=-\dfrac{1}{20\varepsilon}$ and $Y=[Y_1,Y_2,\ldots,Y_{N-1}]^T$, $T(h)=[T_1,T_2,\ldots,T_{N-1}]^T$, $O=[0,0,\ldots,0]^T$ are associated vectors of Eq. (9).

Let $y=[y_1, y_2, \ldots\ldots, y_{N-1}]^T \cong Y$ satisfy the equation

$$(D+P)y+Q=0. \tag{10}$$

Let $e_i = y_i - Y_i$, $i=1,2,\ldots,N-1$ be the discretisation error so that $E=[e_1,e_2,\ldots,e_{N-1}]^T = y-Y$. Using Eq.(9), from Eq.(10) we obtain the error equation:

$$(D+P)E=T(h)\ . \tag{11}$$

Let $|p(x)|\le C_1$ and $|q(x)|\le C_2$, where C_1, C_2 are positive constants. If $P_{i,j}$ is the $(i,j)^{th}$ element of P, then

$$|P_{i,i+1}| = 1+\frac{h^2\alpha}{\varepsilon-k}C_2 \ne 0 \ \text{ for } \ i=1,$$

$$|P_{i,i+1}| = |w_i| \le \left(h(\alpha+\beta)C_1+h^2\alpha C_2+4\beta\omega h^2 C_1^2+2h^3\beta\omega C_1 C_2\right), \qquad i=2,3,\ldots,N-2,$$

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$$\left|P_{i,i-1}\right| = \left|z_i\right| \le \left(h(\alpha+\beta)C_1 + h^2\alpha C_2 + 4\beta\omega h^2 C_1^2 + 2h^3\beta\omega C_1 C_2\right), \qquad i = 2,3,...,N-2.$$

Thus, for sufficiently small h, we have

$$\left|P_{i,i+1}\right| < \varepsilon, \quad i = 1,2,....,N-2,$$

$$\text{and} \qquad \left|P_{i,i-1}\right| < \varepsilon, \quad i = 2,3,....,N-1. \tag{12}$$

Hence $(D + P)$ is irreducible [13].

Let S_i be the sum of the elements of the i^{th} row of the matrix $(D + P)$; then we have

$$S_i = 1 + \frac{h^2}{\varepsilon - k}\left(2\beta q_1 + \alpha q_2\right) \qquad \text{for } i = 1,$$

$$S_i = h^2\left(\alpha q_{i-1} + 2\beta q_i + \alpha q_{i+1}\right) + 2h^3\beta p_i \omega\left(q_{i+1} - q_{i-1}\right) \qquad \text{for } i = 2,3,...,N-2,$$

$$S_i = \varepsilon + \frac{\alpha h}{2}\left(p_{i-1} - 3p_{i+1}\right) - h\beta p_i + h^2\left(\alpha q_{i-1} + 2\beta q_i\right) - h^2\beta\omega p_i\left(p_{i+1} + p_{i-1}\right) - 2h^3\beta\omega p_i q_{i-1} \quad \text{for } i = N-1.$$

Let $C_{1^*} = \min_{1\le i\le N}\left|p(x)\right|$, $C_1^* = \max_{1\le i\le N}\left|p(x)\right|$ and $C_{2^*} = \min_{1\le i\le N}\left|q(x)\right|$, $C_2^* = \max_{1\le i\le N}\left|q(x)\right|$. Since $0 < \varepsilon << 1$ and $\varepsilon \propto O(h)$, it is possible or easy to verify that for small h, $\left(D+P\right)$ is monotone [13, 14]. Hence $\left(D+P\right)^{-1}$ exists and $\left(D+P\right)^{-1} \ge 0$. Thus, using Eq. (10) we have

$$\left\|E\right\| \le \left\|\left(D+P\right)^{-1}\right\| \; \left\|T\right\|. \tag{13}$$

Let $\left(D+P\right)_{i,k}^{-1}$ be the $(i,k)^{th}$ element of $\left(D+P\right)^{-1}$ and we define

$$\left\|\left(D+P\right)^{-1}\right\| = \max_{1\le i\le N-1}\sum_{k=1}^{N-1}\left(D+P\right)_{i,k}^{-1} \quad \text{and} \quad \left\|T(h)\right\| = \max_{1\le i\le N-1}\left|T(h)\right|. \tag{14a}$$

Since $\left(D+P\right)_{i,k}^{-1} \ge 0$ and $\sum_{k=1}^{N-1}\left(D+P\right)_{i,k}^{-1} S_k = 1$ for $i = 1,2,...,N-1$,

hence

$$\left(D+P\right)_{i,k}^{-1} \le \frac{1}{S_i} < \frac{\varepsilon - k}{h^2\left(\alpha + 2\beta\right)C_{2^*}}, \quad i = 1, \tag{14b}$$

$$\left(D+P\right)_{i,k}^{-1} \le \frac{1}{S_i} < \frac{1}{h^2\left[\left(\alpha+2\beta\right)C_{2^*} - 4\beta\omega C_{1^*}^2\right]}, \quad i = N-1. \tag{14c}$$

Furthermore,

$$\sum_{k=1}^{N-1}\left(D+P\right)_{i,k}^{-1} \le \frac{1}{\min_{2\le i\le N-2} S_i} < \frac{1}{h^2\left(2\left(\alpha+\beta\right)C_{2^*}\right)}. \tag{14d}$$

Using Eq.(14a) - (14d), from Eq.(13) we get

$$\left\|E\right\| \le O(h^4). \tag{15}$$

Hence the method (Eq. 7) is fourth-order convergent for $\alpha = \frac{1}{12}$, $\beta = \frac{5}{12}$, $\omega = -\frac{1}{20\varepsilon}$.

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NUMERICAL EXAMPLES

To demonstrate the proposed method computationally, we consider three problems of the type in Eq.(1). These problems are chosen because they have been widely discussed in the literature and exact solutions are also available for comparison.

Example 1. Consider a singularly perturbed singular boundary value problem:

$$-\varepsilon y'' + (1/x)y' + (1+x^2)y = f(x), \quad 0 < x < 1.$$

The exact solution is $y(x) = \exp(x^2)$. The maximum absolute errors are tabulated in Table 1 for different values of ε and h. A comparison of the root mean square errors with another method is presented in Table 2.

Example 2. Consider a boundary value problem:

$$-\varepsilon y'' + \frac{1}{x} y' = f(x), \quad 0 < x < 1.$$

The exact solution of this problem is $y(x) = x \ \sinh x$. The maximum absolute errors are presented in Table 1 for different values of ε and h. A comparison of the root mean square errors is presented in Table 3.

Table 1. Maximum absolute errors in solutions of Examples 1-3

ε / h	2^{-3}	2^{-4}	2^{-5}	2^{-6}
		Example 1		
2^{-7}	5.4581(-3)	6.2584(-4)	5.4501(-5)	4.4412(-6)
2^{-8}	5.7610(-3)	8.8967(-4)	8.7538(-5)	7.4555(-6)
2^{-9}	5.8032(-3)	1.0966(-3)	1.3843(-4)	1.2483(-5)
2^{-10}	5.8522(-3)	1.1948(-3)	2.0262(-4)	2.0646(-5)
		Example 2		
2^{-7}	1.9167(-3)	2.6578(-4)	2.5005(-5)	2.0180(-6)
2^{-8}	1.9443(-3)	3.5059(-4)	3.9742(-5)	3.4527(-6)
2^{-9}	1.9099(-3)	3.9825(-4)	5.9784(-5)	5.8019(-6)
2^{-10}	1.8788(-3)	4.0779(-4)	8.0566(-5)	9.4298(-6)
		Example 3		
2^{-7}	1.9358(-2)	4.9034 (-3)	1.0762(-3)	6.0727(-4)
2^{-8}	1.9788(-2)	5.2394 (-3)	1.2797 (-3)	2.7586 (-4)
2^{-9}	1.9869(-3)	5.3754(-3)	1.3508(-3)	3.2685(-4)
2^{-10}	1.9895(-3)	5.4174(-3)	1.3774(-3)	3.4239(-4)

Table 2. Comparison of root mean square errors in solution of Example 1

ε/N	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Mohanty-Arora method [10]								
2^{-3}	3.94(-2)	1.69(-2)	9.83(-3)	6.75(-3)	4.75(-3)	3.33(-3)	2.37(-3)	1.67(-3)
2^{-4}	6.15(-2)	2.45(-2)	1.36(-2)	9.35(-3)	6.59(-3)	4.65(-3)	3.28(-3)	2.32(-3)
2^{-5}	9.21(-2)	3.29(-2)	1.70(-2)	1.15(-2)	8.10(-3)	5.72(-3)	4.04(-3)	2.85(-3)
2^{-7}	-	4.37(-2)	2.03(-2)	1.49(-2)	1.06(-2)	7.52(-3)	5.31(-3)	3.75(-3)
2^{-8}	-	4.63(-2)	4.05(-2)	3.86(-2)	2.82(-2)	1.99(-2)	1.40(-2)	9.94(-3)
2^{-9}	-	8.15(-2)	7.81(-2)	7.34(-2)	6.93(-2)	4.89(-2)	3.46(-2)	2.44(-2)
2^{-10}	-	8.38(-2)	8.01(-2)	7.77(-2)	7.22(-2)	6.65(-2)	5.87(-2)	4.88(-2)
Proposed method								
2^{-3}	1.80(-4)	1.25(-4)	8.95(-6)	6.30(-7)	4.38(-8)	3.02(-9)	2.07(-10)	1.47(-11)
2^{-4}	2.20(-3)	1.66(-4)	1.24(-5)	9.11(-7)	6.55(-8)	4.63(-9)	3.24(-10)	2.18(-11)
2^{-5}	2.80(-3)	2.33(-4)	1.84(-5)	1.40(-6)	1.04(-7)	7.60(-9)	5.42(-10)	3.82(-11)
2^{-7}	4.00(-3)	4.80(-4)	4.43(-5)	3.72(-6)	2.95(-7)	2.25(-8)	1.66(-09)	1.20(-10)
2^{-8}	4.24(-3)	6.49(-4)	6.87(-5)	6.13(-6)	5.05(-7)	3.95(-8)	2.98(-09)	2.19(-10)
2^{-9}	4.27(-3)	7.79(-4)	1.03(-4)	1.00(-5)	8.62(-7)	6.94(-8)	5.35(-09)	3.99(-10)
2^{-10}	4.30(-3)	8.39(-4)	1.44(-4)	1.60(-5)	1.45(-6)	1.21(-7)	9.57(-09)	7.26(-10)

Table 3. Comparison of root mean square errors in solution of Example 2

ε/N	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
Mohanty-Arora method [10]								
2^{-3}	3.31(-4)	2.18(-3)	1.55(-4)	1.09(-4)	7.68(-5)	5.42(-5)	3.83(-5)	2.70(-5)
2^{-4}	7.42(-4)	5.30(-4)	3.71(-4)	2.60(-4)	1.83(-4)	1.29(-4)	9.14(-5)	6.46(-5)
2^{-5}	1.22(-3)	8.28(-4)	5.71(-4)	4.01(-4)	2.82(-4)	1.99(-4)	1.40(-4)	9.95(-5)
2^{-6}	1.64(-3)	1.06(-3)	7.27(-4)	5.09(-4)	3.59(-4)	2.53(-4)	1.79(-4)	1.26(-4)
2^{-7}	1.95(-3)	1.22(-3)	8.33(-4)	5.84(-4)	4.11(-4)	2.90(-4)	2.05(-4)	1.44(-4)
2^{-8}	2.17(-3)	1.33(-3)	9.01(-4)	6.31(-4)	4.44(-4)	3.14(-4)	2.21(-4)	1.56(-4)
2^{-9}	2.32(-3)	1.40(-3)	9.42(-4)	6.60(-4)	4.65(-4)	3.28(-4)	2.31(-4)	1.63(-4)
2^{-10}	2.43(-3)	1.44(-3)	9.66(-4)	6.77(-4)	4.77(-4)	3.36(-4)	2.37(-4)	1.68(-4)
Proposed method								
2^{-3}	6.48(-4)	4.73(-5)	3.39(-6)	2.39(-7)	1.67(-8)	1.15(-9)	7.89(-11)	5.29(-12)
2^{-4}	8.22(-4)	6.42(-5)	4.84(-6)	3.56(-7)	2.56(-8)	1.81(-9)	1.26(-10)	8.75(-12)
2^{-5}	1.10(-3)	9.19(-5)	7.34(-6)	5.62(-7)	4.18(-8)	3.03(-9)	2.16(-10)	1.51(-11)

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Table 3. (Continued)

ε / N	2^3	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}
2^{-7}	1.40(-3)	1.88(-4)	1.79(-5)	1.52(-6)	1.20(-7)	9.18(-9)	6.78(-10)	4.89(-11)
2^{-8}	1.42(-3)	2.46(-4)	2.77(-5)	2.51(-6)	2.07(-7)	1.62(-8)	1.22(-9)	8.93(-11)
2^{-9}	1.40(-3)	2.80(-4)	4.10(-5)	4.10(-6)	3.55(-7)	2.86(-8)	2.19(-9)	1.63(-10)
2^{-10}	1.38(-3)	2.88(-4)	5.53(-5)	6.50(-6)	6.00(-7)	5.04(-8)	3.94(-09)	2.98(-10)

Example 3. Consider a boundary value problem:

$$\varepsilon y'' + \frac{1}{x} y' + y = 0, \quad 0 < x < 1,$$

with boundary conditions $y(0) = 0,\ y(1) = \exp\left(\dfrac{-1}{2}\right)$, whose exact solution is not known. The maximum absolute errors for this example are calculated by using the double mesh principle $E^N = \max_{0 \le i \le N} \left| y_i^N - y_{2i}^{2N} \right|$ and tabulated in Table 1 for different values of ε and h.

CONCLUSIONS

In this paper the non-polynomial spline method is discussed for a class of singularly perturbed singular two-point boundary value problems. Convergence of the numerical method is analysed. The maximum absolute errors in the solution are tabulated for the existing standard examples chosen from the literature with a view to demonstrating the method. Root mean square errors in the solution of the examples are presented with comparison in order to justify the method. Based on the numerical results, it is observed that the method also affords good results for smaller values of ε. The proposed method is also extendable to non-singular problems and singularly perturbed delay differential equations.

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