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Full Paper

Some criteria on invariant values

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Abstract: Let $K = \mathrm{Q}(\sqrt{d})$ be a real quadratic field, where d is a positive square-free integer congruent to 1 modulo 4. In this paper we determine the first term of the symmetric part of the continued fraction expansion of the integral basis element $\omega_d = (1+\sqrt{d})/2$. Furthermore, we obtain a necessary and sufficient condition for Yokoi's d-invariant value to be zero and present some numerical results on the class number of such real quadratic fields.

Keywords: invariant value, continued fraction, class number, real quadratic field

INTRODUCTION

Let $K = \mathbb{Q}(\sqrt{d})$ be a real quadratic number field, where d is a positive square-free integer. The continued fraction expansion of the integral basis element ω_d in $\mathbb{Q}(\sqrt{d})$ is denoted by $\omega_d = [a_0, \overline{a_1, \dots, a_{k_d-1}, 2a_0-1}]$, where $a_1, a_2, \dots, a_{k_d-1}$ are partial quotients and k_d is the period length of ω_d. The fundamental unit ε_d of the real quadratic number field is also denoted by

$$\varepsilon_d = \left(\frac{T_d + U_d\sqrt{d}}{\sigma}\right) > 1, \text{ where } \sigma = \begin{cases} 2 & if\ d \equiv 1\ mod(4) \\ 1 & if\ d \equiv 2,3\ mod(4) \end{cases}.$$

For any square-free integer d, Yokoi [1] defined some new invariants by taking the fundamental unit of $\mathbb{Q}(\sqrt{d})$ as

$$m_d = \left[\!\left[\frac{{U_d}^2}{T_d}\right]\!\right] \text{ and } n_d = \left[\!\left[\frac{T_d}{{U_d}^2}\right]\!\right],$$

where $[\![x]\!]$ represents the greatest integer less than or equal to x. Yokoi also studied the relationship between these new invariants and the class number of real quadratic fields.

The class-number-one problem for real quadratic fields is still a mysterious classical problem. It is also essential to determine the fundamental units and Yokoi's invariants in order to examine the structures of real quadratic fields and study the class number problems. In an attempt to identify Yokoi's invariants and fundamental units, many studies have been made using various methods. Mollin and Williams [2] proved that if $n_d \neq 0$, then $\varepsilon_d < 8d/\sigma^2$ and $h_d \geq \log n_d / \log p_d$, where p_d is the least prime which splits in $\mathbb{Q}(\sqrt{d})$. Also, Yokoi [3] provided the following bounds for fundamental units and class numbers when the norm of the fundamental unit is equal to -1 :

$$\varepsilon_d > \frac{d-4}{n_d+1} \quad and \quad h_d < \frac{\frac{1}{4}\sqrt{d}(2+\log d)}{\log\left(\frac{d-4}{n_d+1}\right)} .$$

Furthermore, Yokoi [4] investigated the Diophantine equation $x^2 - Dy^2 = \pm 2$ and provided sufficient conditions for the solvability in terms of Yokoi's d-invariants.

On the other hand, some mathematicians have studied real quadratic fields which depend on the period length of the continued fraction expansion of ω_d. Tomita [5, 6] investigated the fundamental units of real quadratic fields $\mathbb{Q}(\sqrt{d})$, where d is a positive square-free integer congruent to 1 modulo 4, and described explicitly the form of fundamental units such that period k_d is equal to 3, 4 and 5. He also provided some criteria on Yokoi's d-invariants using these explicit forms of fundamental units. Similarly, it was determined that the form of fundamental units for k_d is equal to $6, 7$ or 8 [7-10]. However, it is observed that the research on Yokoi's invariant for $k_d > 6$ does not exist. This study aims to obtain a necessary and sufficient condition for Yokoi's d-invariant n_d to be zero with period $k_d = 8$, using the explicit form of the fundamental unit in the literature [9]. It also aims to generalise the condition so that it is valid for all periods using a different method. This study helps in improving estimates of fundamental units and class numbers for real quadratic fields. It also provides conditions for the solvability of some Diophantine equations.

PRELIMINARIES

In this section some of the important preliminaries and lemmas are given. For any square-free positive integer d, we can put $d = a^2 + b$ with $a, b \in \mathbb{Z}, 0 < b \leq 2a$. Here, since $\sqrt{d} - 1 < a < \sqrt{d}$, the integers a and b are uniquely determined by d. Let $d = a^2 + b$ be a square-free positive integer congruent to 1 modulo 4; then we consider the following two cases in which a is even or odd respectively:

Case 1. If a is even, then $b = 4m + 1$ with $m \in \mathbb{Z}, m \geq 0$.

Case 2. If a is odd, then $b = 4m$ with $m \in \mathbb{Z}, m \geq 1$.

Tomita [6] has defined set D_t^s by

$$D_t^s = \{d \in \mathbb{Z}^+ \mid d \equiv s \; mod(8), \; b \equiv t \; mod(8)\},$$

where $\mathbb{Z}^+$ is the set of all positive integers.

Remark 1. The integer d can be congruent to 1 or 5 modulo 8 when d is congruent to 1 modulo 4.

In the case of $d = a^2 + b \equiv 1 \; mod(8)$, b can be congruent to 0, 1 or 5 modulo 8. Therefore, the set of all positive square-free integers congruent to 1 modulo 8 is equal to $D_0^1 \cup D_1^1 \cup D_5^1$.

In the case of $d = a^2 + b \equiv 5 \; mod(8)$, b can be congruent to 1, 4 or 5 modulo 8. Therefore, the set of all positive square-free integers congruent to 5 modulo 8 is equal to $D_1^5 \cup D_4^5 \cup D_5^5$.

Remark 2. Let $d = a^2 + b$ be a square-free positive integer congruent to 1 modulo 4.

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If a is even, then $b \equiv 1\ mod(4)$. Hence b can only be congruent to 1 or 5 modulo 8. Therefore, d belongs to $D_1^5 \cup D_5^5 \cup D_5^1 \cup D_1^1$ in the case in which a is even.

If a is odd, then $b \equiv 0\ mod(4)$. Hence b can only be congruent to 0 or 4 modulo 8. Therefore, d belongs to $D_0^1 \cup D_4^5$ in the case in which a is odd.

Remark 3. The sets $D_0^1, D_1^1, D_5^1, D_1^5, D_4^5$ and D_5^5 are represented as:

$D_0^1 = \{d \in D \mid d = a^2 + 8m,\ a \equiv 1\ mod(2),\ 0 < 4m < a\}$,
$D_1^1 = \{d \in D \mid d = a^2 + 8m + 1,\ a \equiv 0\ mod(4),\ 0 \leq 4m < a\}$,
$D_5^1 = \{d \in D \mid d = a^2 + 8m + 5,\ a \equiv 2\ mod(4),\ 0 \leq 4m < a - 2\}$,
$D_1^5 = \{d \in D \mid d = a^2 + 8m + 1,\ a \equiv 2\ mod(4),\ 0 \leq 4m < a\}$,
$D_4^5 = \{d \in D \mid d = a^2 + 8m + 4,\ a \equiv 1\ mod(2),\ 0 \leq 4m < a - 2\}$,
$D_5^5 = \{d \in D \mid d = a^2 + 8m + 5,\ a \equiv 0\ mod(4),\ 0 \leq 4m < a - 2\}$.

Let $I(d)$ be the set of all quadratic irrational numbers in $\mathbb{Q}(\sqrt{d})$. For an element ξ of $I(d)$, if $\xi > 1$ and $-1 < \xi' < 0$, then ξ is called reduced; ξ' is the conjugate of ξ with respect to $\mathbb{Q}$. The set of all reduced quadratic irrational numbers in $I(d)$ is denoted by $R(d)$. It is well known that if an element ξ of $I(d)$ is in $R(d)$, then the continued fractional expansion of ξ is purely periodic. Moreover, the denominator of its modular automorphism is equal to fundamental unit ε_d of $\mathbb{Q}(\sqrt{d})$.

In order to prove our theorems, we need the following lemmas. We will write k instead of k_d for convenience.

Lemma 1 [5]. For a square-free positive integer $d > 5$ congruent to 1 modulo 4, we put $\omega_d = \frac{1+\sqrt{d}}{2}$, $q_0 = [\![\omega_d]\!]$, $\omega_R = q_0 - 1 + \omega_d$. Then $\omega_d \notin R(d)$, but $\omega_R \in R(d)$ holds. Moreover, for the period k of ω_R, we get $\omega_R = [\,\overline{2q_0 - 1,\ q_1, \dots, q_{k-1}}\,]$ and $\omega_d = [q_0, \overline{q_1, \dots, q_{k-1}, 2q_0 - 1}]$. Furthermore, let $\omega_R = \frac{P_k\omega_R + P_{k-1}}{Q_k\omega_R + Q_k} = [2q_0 - 1, q_1, \dots, q_{k-1}, \omega_R]$ be a modular automorphism of ω_R; then the fundamental unit ε_d of $\mathbb{Q}(\sqrt{d})$ is given by the following formula:

$$\varepsilon_d = \left(\frac{T_d + U_d\sqrt{d}}{2}\right) > 1,$$
$$T_d = (2q_0 - 1)Q_k + 2Q_{k-1} \text{ and } U_d = Q_k$$

where Q_i is determined by $Q_0 = 0,\ Q_1 = 1,\ Q_{i+1} = q_iQ_i + Q_{i-1}, (i \geq 1)$.

Lemma 2 [11]. For a square-free positive integer d, we put $d = a^2 + b\ \ (0 < b \leq 2a), a, b \in \mathbb{Z}$. Moreover, let $\omega_i = \ell_i + \frac{1}{\omega_{i+1}}$ $(\ell_i = [\![\omega_i]\!],\ i \geq 0)$ be the continued fraction expansion of $\omega = \omega_0$ in $R(d)$. Then each ω_i is expressed in the form $\omega_i = (a - r_i + \sqrt{d})/c_i$, $(c_i, r_i \in \mathbb{Z})$, and ℓ_i, c_i, r_i can be obtained from the following recurrence formula:

$$\omega_0 = (a - r_0 + \sqrt{d})/c_0,$$
$$2a - r_i = c_i\ell_i + r_{i+1},$$
$$c_{i+1} = c_{i-1} + (r_{i+1} - r_i)\ell_i\ \ (i \geq 0), where\ 0 \leq r_{i+1} < c_i,$$
$$c_{-1} = (b + 2ar_0 + {r_0}^2)/c_0.$$

Moreover, for the period $k \geq 1$ of ω_0, we get

$$\ell_i = \ell_{k-i}\ \ (1 \leq i \leq k - 1),$$
$$r_i = r_{k-i+1},\ \ c_i = c_{k-i}\ (1 \leq i \leq k).$$

Lemma 3 [5]**.** For a square-free positive integer d congruent to 1 modulo 4, we put $\omega_d = \frac{1+\sqrt{d}}{2}$, $q_0 = [\![\omega_d]\!]$, $\omega_R = q_0 - 1 + \omega_d$. If we put $\omega = \omega_R$ in Lemma 2, then we have the following recurrence formula:

$$r_0 = r_1 = a - \ell_0 = a - 2q_0 + 1\,,$$
$$c_0 = 2, \quad c_1 = c_{-1} = (b + 2ar_0 + {r_0}^2)/c_0\,,$$
$$\ell_0 = 2q_0 - 1\,, \quad \ell_i = q_i \quad (1 \le i \le k-1).$$

Lemma 4 [9]**.** Let $d = a^2 + b \equiv 1\, mod(4)$ be a square-free integer for positive integers a and b satisfying $0 < b \le 2a$. Let the period k_d of the integral basis element of $\omega_d = \frac{1+\sqrt{d}}{2}$ in $\mathbb{Q}(\sqrt{d})$ be 8. If a is odd, then

$$\omega_d = \left[\frac{a+1}{2}, \overline{\ell_1, \ell_2, \ell_3, \frac{s_1(C+\ell_3 A) - 2\ell_2 B - rA}{C(r - s_1\ell_3) + B^2}, \ell_3, \ell_2, \ell_1, a}\right],$$

where $\ell_i \ge 1$ $(i = 1, 2, 3, 4)$. Then the coefficients T_d and U_d of ε_d, where

$$T_d = (Ar + s_1\ell_1)(C^2\ell_4 + 2AC) + 2(C(B\ell_4 + \ell_2) + B), \quad U_d = C(C\ell_4 + 2A)$$

and

$$d = (Ar + s_1\ell_1)^2 + 4r\ell_2 + 4s_1\,,$$

hold. A, B, C are determined by $A = \ell_1\ell_2 + 1$, $B = \ell_2\ell_3 + 1$ and $C = \ell_1 + A\ell_3$. Moreover, r and s are uniquely determined with the equalities $a = Ar + s_1\ell_1$ and

$$B(B\ell_4 + 2\ell_2) = s_1[C(1 + \ell_3\ell_4) + A\ell_3] - r(A + C\ell_4).$$

Lemma 5 [9]**.** Let $d = a^2 + b \equiv 1\, mod(4)$ be a square-free integer for positive integers a and b satisfying $0 < b \le 2a$. Let the period k_d of the integral basis element of $\omega_d = \frac{1+\sqrt{d}}{2}$ in $\mathbb{Q}(\sqrt{d})$ be 8. If a is even, then

$$\omega_d = \left[\frac{a}{2}, \overline{\ell_1, \ell_2, \ell_3, \frac{BC + AD - 2\ell_3}{{\ell_3}^2 - CD}, \ell_3, \ell_2, \ell_1, a-1}\right], \qquad \ell_i \ge 1\ (i = 2{,}3)$$

and then the coefficients T_d and U_d of ε_d, where

$$T_d = [(A(r+1) + B - 2)C + 2(C - \ell_3)](C\ell_4 + 2A) + 2C\ell_2\,,$$
$$U_d = C(C\ell_4 + 2A)$$

and

$$d = [A(r+1) + B - 1]^2 + 2[A(r+1) + B - 2s - 2] - 1\,,$$

hold. $A = \ell_2 + 1$, $B = 2s - r$, $C = 1 + A\ell_3$, $D = B\ell_3 - r - 1$, $E = \ell_3 + 1$, and r and s are uniquely determined with the equalitions $a = A(r+1) + B - 1$ and

$$\ell_3(\ell_3\ell_4 + 2) = BC + AD + 2CD\ell_4.$$

MAIN RESULTS

Theorem 1. Let $d = a^2 + b \equiv 1\, mod(4)$ be a square-free integer for positive integers a and b satisfying $0 < b \le 2a$. Let the period of the continued fraction expansion of the integral basis element of $\omega_d = \frac{1+\sqrt{d}}{2}$ in $\mathbb{Q}(\sqrt{d})$ be denoted by k_d. For every $k_d > 1$,

i) If a is odd, then $\ell_1 \ne 1$, i.e. $\ell_1 \ge 2$;
ii) If a is even, then $\ell_1 = 1$.

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Proof. If $d = a^2 + b \equiv 1 \, mod(4)$, then $d \in D_0^1 \cup D_4^5 \cup D_1^1 \cup D_1^5 \cup D_5^5 \cup D_5^1$.

i) Let a be odd. Then $q_0 = [\![\omega_d]\!] = \left[\!\!\left[\frac{1+\sqrt{d}}{2}\right]\!\!\right] = \frac{a+1}{2}$ and the continued fraction expansion of the integral basis element of ω_d in $\mathbb{Q}(\sqrt{d})$ is

$$\omega_d = \left[\frac{a+1}{2}, \overline{\ell_1, \ell_2, \ell_3, \dots, \ell_3, \ell_2, \ell_1, a}\right], \ \ell_i \geq 1 \ (i = 1,2,3, \dots).$$

From Lemma 3 it is obtained that

$$r_0 = r_1 = a - 2q_0 + 1 = 0,$$
$$c_0 = 2,$$
$$c_1 = c_{-1} = (b + 2ar_0 + {r_0}^2)/c_0 = b/2,$$
$$\ell_0 = 2q_0 - 1 = a.$$

Since a is odd, d belongs to $D_0^1 \cup D_4^5$.

We first assume that $d \in D_0^1 = \{d \in D : d = a^2 + 8m, a \equiv 1 \, mod(2), 0 < 4m < a\}$. Then $b = 8m$ and so $c_{-1} = c_1 = 4m$. For $i = 1$ in Lemma 2, it is obtained that

$$r_2 = 2a - 4m\ell_1.$$

Using $0 \leq r_2 < c_1$, it can be written that $a < 2m(1 + \ell_1)$. On the other hand, we have $0 < 4m < a$ from the assumption. Hence the following inequality is clear:

$$0 < 4m < a < 2m(1 + \ell_1).$$

If we write $\ell_1 = 1$ in the last inequality, then we get a contradiction. Thus, $l_1 \neq 1$ is obtained.

Now we assume that $d \in D_4^5 = \{d \in D : d = a^2 + 8m + 4, \ a \equiv 1 \, mod(2), \ 0 \leq 4m < a - 2\}$. Then $b = 8m + 4$ and so $c_{-1} = c_1 = 4m + 2$. For $i = 1$ in Lemma 2, it is obtained that

$$r_2 = 2a - (4m + 2)\ell_1 .$$

Using $0 \leq r_2 < c_1$, it can be written that

$$a - 2 < 2m + 2m\ell_1 + \ell_1 - 1.$$

On the other hand, we have $0 \leq 4m < a - 2$ from the assumption. Hence the following inequality is clear:

$$2m < 2m\ell_1 + \ell_1 - 1.$$

If we write $\ell_1 = 1$ in the last inequality, then we get a contradiction. Thus, $\ell_1 \neq 1$ is obtained. Therefore, the first part of proof is completed.

ii) Let a be even. Then $q_0 = [\![\omega_d]\!] = \left[\!\!\left[\frac{1+\sqrt{d}}{2}\right]\!\!\right] = \frac{a}{2}$ and the continued fraction expansion of the integral basis element of ω_d in $\mathbb{Q}(\sqrt{d})$ is

$$\omega_d = \left[\frac{a}{2}, \overline{\ell_1, \ell_2, \ell_3, \dots, \ell_3, \ell_2, \ell_1, a - 1}\right], \ \ell_i \geq 1 \ (i = 1,2,3, \dots).$$

From Lemma 3 it is obtained that

$$r_0 = r_1 = a - 2q_0 + 1 = 1,$$
$$c_0 = 2,$$
$$c_1 = c_{-1} = (b + 2ar_0 + {r_0}^2)/c_0 = (b + 2a + 1)/2,$$
$$\ell_0 = 2q_0 - 1 = a - 1.$$

Since a is even, d belongs to $D_1^1 \cup D_1^5 \cup D_5^5 \cup D_5^1$.

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We assume that $d \in D_1^1 \cup D_1^5$; then $b = 8m + 1$ $(m \geq 0)$ and so $c_{-1} = c_1 = 4m + a + 1$. For $i = 1$ in Lemma 2, it is obtained that $a(2 - \ell_1) = 4m\ell_1 + \ell_1 + r_2 + 1$, which implies that $2 > \ell_1$. Since $2 > \ell_1$ and $\ell_1 \geq 1$, we get $\ell_1 = 1$.

We assume that $d \in D_5^5 \cup D_5^1$; then $b = 8m + 5$ $(m \geq 0)$ and so $c_{-1} = c_1 = 4m + a + 3$. For $i = 1$ in Lemma 2, it is obtained that $a(2 - \ell_1) = 4m\ell_1 + 3\ell_1 + r_2 + 1$, which implies that $2 > \ell_1$. Similarly, we get $\ell_1 = 1$. Therefore, the proof of Theorem 1 is completed.

Theorem 2. Suppose that $d = a^2 + b \equiv 1\, mod(4)$ is a square-free integer for positive integers a and b satisfying $0 < b \leq 2a$. Let the period k_d of the integral basis element of $\omega_d = \frac{1+\sqrt{d}}{2}$ in $\mathbb{Q}(\sqrt{d})$ be 8. In the case that a is odd,

$$n_d = 0 \; if \; and \; only \; if \; U_d > a\,.$$

Proof. Note that $n_d = \left[\!\left[\frac{T_d}{{U_d}^2}\right]\!\right] = 0$ if and only if ${U_d}^2 - T_d > 0$. Now let $n_d = 0$. Using coefficients T_d and U_d in Lemma 4, the following can be written:

$${U_d}^2 - T_d = U_d(U_d - a) - 2[BC\ell_4 + C\ell_2 + B] > 0.$$

Thus, $U_d - a$ must be a positive integer. Therefore, we get $U_d > a$.

Conversely, suppose that $U_d > a$. We know that $\ell_1 \neq 1$ from Theorem 1 in the case that a is odd. We define $U_d - a = K$. Since $U_d > a$, we get $U_d - a = K > 0$. It follows that

$${U_d}^2 - T_d = U_d(U_d - a) - 2[BC\ell_4 + C\ell_2 + B] = C\ell_4[KC - 2B] + 2[KAC - C\ell_2 - B].$$

Now we will show that $KC - 2B > 0$ and $KAC - C\ell_2 - B > 0$. If we put $A = \ell_1\ell_2 + 1$, $B = \ell_2\ell_3 + 1$ and $C = \ell_1 + A\ell_3$, then we get

$$\begin{aligned} KC - 2B &= K(\ell_1 + \ell_3 + \ell_1\ell_2\ell_3) - 2\ell_2\ell_3 - 2 \\ &= \ell_2\ell_3(K\ell_1 - 2) + (K\ell_1 - 2) + K\ell_3. \end{aligned}$$

Since $K > 0$ and $\ell_1 \neq 1$, it is obtained that $K\ell_1 - 2 \geq 0$, and so $KC - 2B > 0$. On the other hand, the following can be written:

$$\begin{aligned} KAC - C\ell_2 - B &= KC(\ell_1\ell_2 + 1) - C\ell_2 - \ell_2\ell_3 - 1 \\ &= C\ell_2(K\ell_1 - 1) + (K\ell_1 - 1) + \ell_2\ell_3(K\ell_1 - 1) + K\ell_3. \end{aligned}$$

Since $K > 0$ and $\ell_1 \neq 1$, then it is obtained that $K\ell_1 - 1 > 0$. Therefore, ${U_d}^2 - T_d > 0$ holds and this finishes the proof.

Theorem 3. Suppose that $d = a^2 + b \equiv 1\, mod(4)$ is a square-free integer for positive integers a and b satisfying $0 < b \leq 2a$. Let the period k_d of the integral basis element of $\omega_d = \frac{1+\sqrt{d}}{2}$ in $\mathbb{Q}(\sqrt{d})$ be 8. In the case that a is even,

i) If $n_d = 0$, then $U_d > a$;
ii) If $U_d > a$ and $\ell_3\ell_4 \geq \ell_2$, then $n_d = 0$.

Proof. i) Let $n_d = 0$, that is ${U_d}^2 - T_d > 0$. Using coefficients T_d and U_d in Lemma 5, the following can be written:

$${U_d}^2 - T_d = {U_d}^2 - aU_d - U_d + 2\ell_3(C\ell_4 + 2A) - 2C\ell_2 > 0.$$

Define $L = -U_d + 2\ell_3(C\ell_4 + 2A) - 2C\ell_2$. Then

$$\begin{aligned} L &= (2\ell_3 - C)(C\ell_4 + 2A) - 2C\ell_2 \\ &= (2 - 2A)\ell_3\ell_4 + (2 - A)A\ell_3^2\ell_4 + (2 - A)2A\ell_3 - \ell_4 - 2A - 2\ell_2 - 2A\ell_2\ell_3 \end{aligned}$$

holds. Since $\ell_2 \geq 1$, we get $A = \ell_2 + 1 \geq 2$. Therefore, $2 - 2A \leq -2$ and so $L < 0$. Hence $U_d - a > 0$ holds since $U_d{}^2 - T_d = U_d(U_d - a) + L > 0$ and $L < 0$. This implies $U_d > a$.

ii) Now suppose that $U_d > a$ and $\ell_3\ell_4 \geq \ell_2$. Since $U_d > a$, we can write $U_d - a - 1 \geq 0$. Define

$$K = \ell_3(C\ell_4 + 2A) - C\ell_2 = C(\ell_3\ell_4 - \ell_2) + 2A\ell_3.$$

Since $\ell_3\ell_4 \geq \ell_2$, we get $K > 0$. Consequently, we obtain $U_d{}^2 - T_d = U_d(U_d - a - 1) + 2K > 0$, i.e. $n_d = 0$. ■

The following theorem is a generalisation of Theorem 2 and Theorem 3.

Theorem 4. Let $d = a^2 + b \equiv 1\, mod(4)$ be a square-free integer for positive integers a and b satisfying $0 < b \leq 2a$. Let the period of the integral basis element of $\omega_d = \frac{1+\sqrt{d}}{2}$ in $\mathbb{Q}(\sqrt{d})$ be denoted by k_d. For every $k_d \geq 1$,

$$n_d = 0 \text{ if and only if } U_d > a.$$

Proof. We examine two cases:

<u>Case I. a is odd:</u>

Suppose that $n_d = 0$, that is $U_d{}^2 - T_d > 0$. We write simply k instead of k_d. Since a is odd, we get $q_0 = [\![\omega_d]\!] = \left[\!\!\left[\frac{1+\sqrt{d}}{2}\right]\!\!\right] = \frac{a+1}{2}$ and $T_d = aQ_k + 2Q_{k-1}$, $U_d = Q_k$ from Lemma 1. Thus, it can be written that

$$U_d{}^2 - T_d = Q_k{}^2 - aQ_k - 2Q_{k-1} = Q_k(Q_k - a) - 2Q_{k-1} > 0.$$

This implies that $U_d = Q_k > a$.

Conversely, let $U_d > a$. By taking equation $Q_{i+1} = q_iQ_i + Q_{i-1}, (i \geq 1)$ in Lemma 1, for $i = k - 1$ the following can be written:

$$Q_k = q_{k-1}Q_{k-1} + Q_{k-2}.$$

From equation $\ell_i = q_i \quad (1 \leq i \leq k - 1)$ in Lemma 3, for $i = k - 1$,

$$\ell_{k-1} = q_{k-1}.$$

In Lemma 2 by writing $i = k - 1$ in equation $\ell_i = \ell_{k-i} \;\; (1 \leq i \leq k - 1)$, we obtain

$$\ell_{k-1} = \ell_{k-k+1} = \ell_1.$$

From Theorem 1 we know that $\ell_1 \geq 2$ when a is odd. Thus,

$$Q_k = q_{k-1}Q_{k-1} + Q_{k-2} = \ell_1 Q_{k-1} + Q_{k-2} \geq 2Q_{k-1} + Q_{k-2},$$

and so $Q_k > 2Q_{k-1}$ holds. Since we assume that $Q_k = U_d > a$,

$$Q_k(Q_k - a) \geq Q_k > 2Q_{k-1} \text{ and } U_d{}^2 - T_d = Q_k{}^2 - aQ_k - 2Q_{k-1} > 0$$

hold. Consequently, $n_d = 0$ is obtained.

<u>Case II. a is even:</u>

Since a is even, we get $q_0 = [\![\omega_d]\!] = \left[\!\!\left[\frac{1+\sqrt{d}}{2}\right]\!\!\right] = \frac{a}{2}$ and $T_d = (a - 1)Q_k + 2Q_{k-1}$, while $U_d = Q_k$ from Lemma 1 and $\ell_1 = 1$ from Theorem 1. By taking equation $Q_{i+1} = q_iQ_i + Q_{i-1}, (i \geq 1)$ in Lemma 1, we obtain

$$Q_k = q_{k-1}Q_{k-1} + Q_{k-2} = \ell_1 Q_{k-1} + Q_{k-2} = Q_{k-1} + Q_{k-2} \text{ for } i = k - 1.$$

Then the following can be written:

$$\begin{aligned} U_d{}^2 - T_d &= Q_k{}^2 - (a - 1)Q_k - 2Q_{k-1} = Q_k(Q_k - a) + Q_k - 2Q_{k-1} \\ &= Q_k(Q_k - a) + Q_{k-2} - Q_{k-1}. \end{aligned}$$

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On the other hand, for $i = k-2$ in Lemma 1,

$$Q_{k-1} = q_{k-2}Q_{k-2} + Q_{k-3} = \ell_2 Q_{k-2} + Q_{k-3}$$

holds. Thus, we get $Q_k > Q_{k-1} > Q_{k-2}$, so $0 > Q_{k-2} - Q_{k-1}$.

Now suppose that $n_d = 0$, that is ${U_d}^2 - T_d = Q_k(Q_k - a) + Q_{k-2} - Q_{k-1} > 0$. This implies that $Q_k = U_d > a$ since $0 > Q_{k-2} - Q_{k-1}$.

Conversely, let $U_d > a$. We will investigate $U_d > a+1$ and $U_d = a+1$ separately. In the case of $Q_k = U_d > a+1$, we can write $Q_k - a + 1 > 2$. Thus,

$$(Q_k - a + 1)Q_k > 2Q_k > 2Q_{k-1}$$

or

$${Q_k}^2 - (a-1)Q_k - 2Q_{k-1} > 0$$

holds. Hence it is obtained that

$${U_d}^2 - T_d = Q_k(Q_k - a) - Q_{k-1} + Q_{k-2} > 0, \text{ i.e. } n_d = 0 .$$

In the case of $Q_k = U_d = a+1$, we have

$${U_d}^2 - T_d = {Q_k}^2 - (a-1)Q_k - 2Q_{k-1} = 2a + 2 - 2Q_{k-1} .$$

Since $Q_k = Q_{k-1} + Q_{k-2}$, we get $a + 1 = Q_k > Q_{k-1}$ and

$${U_d}^2 - T_d = 2a + 2 - 2Q_{k-1} > 2Q_{k-1} - 2Q_{k-1} = 0 .$$

Consequently, this implies that $n_d = 0$. Therefore, the proof is completed.

SOME NUMERICAL RESULTS ON THE CLASS NUMBER

In this section we give some numerical results on the class number h_d of the real quadratic field $\mathbb{Q}(\sqrt{d})$. Let $d = a^2 + b \equiv 1\, mod(4)$ be a square-free integer for positive integers a and b satisfying $0 < b \leq 2a$.

Corollary 1. For any d satisfying $3533 < d < 10^7$, we may confirm that $U_d \leq a$ $(n_d \neq 0)$ implies $h_d > 1$ using computer algebra systems such as Mathematica. For $d \leq 3533$, there exist exactly 38 real quadratic fields $\mathbb{Q}(\sqrt{d})$ satisfying $U_d \leq a$ $(n_d \neq 0)$ and $h_d = 1$. All such d's are listed in Table 1.

Real quadratic fields satisfying $n_d \neq 0$ and $h_d = 1$ have been previously attained in the literature. However, we have obtained the same fields according to $U_d \leq a$ criteria without calculating n_d. When we tabulated d values satisfying $U_d \leq a$ and $h_d = 1$ with periods k_d and coefficients of fundamental units, we have observed the following interesting results:

Corollary 2. If $U_d \leq a$ $(n_d \neq 0)$ and $h_d = 1$, then the period of the continued fraction expansion of ω_d is less than or equal to 7.

Corollary 3. Let $d \equiv 1\, mod(4)$ be a square-free integer. Let k_d be the period of the continued fraction expansion of ω_d. By using computer algebra systems, for $1 < d < 10^5$, it can be seen that

i) If d is a prime integer, then k_d is odd;
ii) If d is a prime integer, then h_d is odd;
iii) If h_d is odd and k_d is odd, then d is a prime integer.

Corollary 4. If k_d is odd and the $d \equiv 1\, mod(4)$ square-free integer is not prime, then h_d is even, i.e. $h_d \neq 1$.

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Table 1. All d's with $U_d \le a$ and $h_d = 1$

	d	Continued fraction of $\omega_d = (1+\sqrt{d})/2$	k_d	a	U_d	T_d	n_d	h_d
*	5	$[1,\overline{1}]$	1	2	1	1	1	1
*	13	$[2,\overline{3}]$	1	3	1	3	3	1
*	17	$[2,\overline{1,1,3}]$	3	4	2	8	2	1
	21	$[2,\overline{1,3}]$	2	4	1	5	5	1
*	29	$[3,\overline{5}]$	1	5	1	5	5	1
*	37	$[3,\overline{1,1,5}]$	3	6	2	12	3	1
*	53	$[4,\overline{7}]$	1	7	1	7	7	1
*	61	$[4,\overline{2,2,7}]$	3	7	5	39	1	1
	69	$[4,\overline{1,1,1,7}]$	4	8	3	25	2	1
	77	$[4,\overline{1,7}]$	2	8	1	9	9	1
	93	$[5,\overline{3,9}]$	2	9	3	29	3	1
*	101	$[5,\overline{1,1,9}]$	3	10	2	20	5	1
*	149	$[6,\overline{1,1,1,1,11}]$	5	12	5	61	2	1
*	173	$[7,\overline{13}]$	1	13	1	13	13	1
*	197	$[7,\overline{1,1,13}]$	3	14	2	28	7	1
	213	$[7,\overline{1,3,1,13}]$	4	14	5	73	2	1
	237	$[8,\overline{5,15}]$	2	15	5	77	3	1
*	269	$[8,\overline{1,2,2,1,15}]$	5	16	10	164	1	1
*	293	$[9,\overline{17}]$	1	17	1	17	17	1
*	317	$[9,\overline{2,2,17}]$	3	17	5	89	3	1
	341	$[9,\overline{1,2,1,2,1,17}]$	6	18	15	277	1	1
	413	$[10,\overline{1,1,1,19}]$	4	20	3	61	6	1
	437	$[10,\overline{1,19}]$	2	20	1	21	21	1
	453	$[11,\overline{7,21}]$	2	21	7	149	3	1
*	461	$[11,\overline{4,4,21}]$	3	21	17	365	1	1
*	557	$[12,\overline{3,3,23}]$	3	23	10	236	2	1
*	677	$[13,\overline{1,1,25}]$	3	26	2	52	13	1
	717	$[13,\overline{1,7,1,25}]$	4	26	9	241	2	1
*	773	$[14,\overline{2,2,27}]$	3	27	5	139	5	1
*	797	$[14,\overline{1,1,1,1,1,1,27}]$	7	28	13	367	2	1
*	1013	$[16,\overline{2,2,2,2,31}]$	5	31	29	923	1	1
	1077	$[16,\overline{1,9,1,31}]$	4	32	11	361	2	1

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Table 1. (Continued)

d	*Continued fraction of* $\omega_d = (1+\sqrt{d})/2$	k_d	a	U_d	T_d	n_d	h_d
1133	$[17, \overline{3, 33}]$	2	33	3	101	11	1
1253	$[18, \overline{5, 35}]$	2	35	5	177	7	1
1757	$[21, \overline{2, 5, 2, 41}]$	4	41	24	1006	1	1
* 1877	$[22, \overline{6, 6, 43}]$	3	43	37	1603	1	1
* 2477	$[25, \overline{2, 1, 1, 2, 49}]$	5	49	13	647	3	1
3533	$[30, \overline{4, 1, 1, 4, 59}]$	5	59	41	2437	1	1

Note:
k_d = period length of ω_d ;
$\varepsilon_d = (T_d + U_d\sqrt{d})/2 > 1$: fundamental unit of $\mathbb{Q}(\sqrt{d})$;
$n_d = [\![T_d/{U_d}^2]\!]$;
h_d = class number of $\mathbb{Q}(\sqrt{d})$;
* denotes prime.

CONCLUSIONS

In this paper we have studied n_d, Yokoi's d-invariant value, in terms of continued fractions where $d = a^2 + b \equiv 1 \ (mod\ 4)$ is a square-free integer and have provided a necessary and sufficient condition for n_d to be zero for every $k_d \geq 1$ period. Furthermore, according to the case of a being odd or even, we have also investigated the first term of the symmetric part of the continued fraction expansion of the integral basis element $\omega_d = (1+\sqrt{d})/2$. These findings could establish more effective bounds on the fundamental unit and class number and contribute to the solvability of some Diophantine equations.

REFERENCES

1. H. Yokoi, "New invariants and class number problem in real quadratic fields", *Nagoya Math. J.*, **1993**, *132*, 175-197.
2. R. A. Mollin and H. C. Williams, "A complete generalization of Yokoi's p-invariants", *Colloq. Math.,* **1992**, *63*, 285-294.
3. H. Yokoi, "The fundamental unit and bounds for class numbers of real quadratic fields", *Nagoya Math. J.*, **1991**, *124*, 181-197.
4. H. Yokoi, "Solvability of the Diophantine equation $x^2 - Dy^2 = \pm 2$ and new invariants for real quadratic fields", *Nagoya Math. J.*, **1994**, *134*, 137-149.
5. K. Tomita, "Explicit representation of fundamental units of some real quadratic fields I", *Proc. Japan Acad. Ser. A Math. Sci.*, **1995**, *71*, 41-43.
6. K. Tomita, "Explicit representation of fundamental units of some real quadratic fields II", *J. Number Theory*, **1997**, *63*, 275-285.
7. A. Pekin and H. İşcan, "Continued fractions of period six and explicit representations of fundamental units of some real quadratic fields", *J. Ind. Math. Soc.*, **2005**, *72*, 184-194.

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8. G. K. Gözeri and A. Pekin, "Explicit form of the fundamental units of certain real quadratic fields", *Eur. J. Pure Appl. Math.*, **2014**, *7*, 55-64.
9. Ö. Özer and A. Pekin, "An algorithm for explicit form of fundamental units of certain real quadratic fields and period eight", *Eur. J. Pure Appl. Math.*, **2015**, *8*, 343-356.
10. Ö. Özer and A. Pekin, "An algorithm for explicit form of fundamental units of certain real quadratic fields", *J. Anal. Number Theory*, **2016**, *4*, 23-27.
11. T. Azuhata, "On the fundamental units and the class numbers of real quadratic fields", *Nagoya Math. J.*, **1984**, *95*, 125-135.