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Full Paper

Some properties of multivalent spiral-like functions

Khalida Inayat Noor [1], Nazar Khan [1, 2, *] and Qazi Zahoor Ahamd [2]

[1] Department of Mathematics, COMSATS Institute of Information Technology Chack Shahzad, Islamabad - 44000, Pakistan
[2] Department of Mathematics, Abbottabad University of Science and Technology, Abbottabad, Pakistan
* Corresponding author, e-mail: nazarmaths@gmail.com

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Abstract: In this paper we introduce certain new subclasses of multivalent spiral-like functions by means of the concept of k-uniformly star-likness and k-uniformly convexity. We prove the inclusion relations, sufficient condition and Fekete-Szegö inequality for these classes of functions. The behaviour of these classes under a certain integral operator is also discussed.

Keywords: spiral-like functions, p-valent functions, conic domain

INTRODUCTION

Let $A(p)$ denote a class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in N = \{1, 2, \ldots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $\mathsf{E} = \{z : |z| < 1\}$. The subclass of $A(p)$, consisting of all analytic functions and having a positive real part in E, is denoted by P. An analytic description of P is given by

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathsf{E}).$$

A function $f \in A$ is said to be the $\beta-$ spiral-like function $\left(\beta \text{ real}, |\beta| < \frac{\pi}{2}\right)$ of order ρ $(0 \le \rho < 1)$, if and only if

$$\operatorname{Re}\left(e^{i\beta} \frac{zf'(z)}{f(z)}\right) > \rho \cos\beta \quad (z \in \mathsf{E}).$$

We denote the class of all $\beta-$spiral-like functions of order ρ by $S^*_\beta(\rho)$ [1, 2].

Furthermore, a function $f \in A$ is called $\beta-$spiral-like convex function (β is real, $|\beta| < \frac{\pi}{2}$) if

Maejo Int. J. Sci. Technol. **2018**, *12*(02), 139-151

$$\operatorname{Re}\left[\cos\beta\left(\frac{zf'(z)}{f'(z)}\right)'+i\sin\beta\frac{zf'(z)}{f(z)}\right]>0 \quad (z\in \mathsf{E}).$$

We denote the subclass of A, consisting of all $\beta-$ spiral-like convex functions, by $\mathrm{C}(\beta)$ and it has been shown [3] that $\mathrm{C}(\beta)\subset \mathrm{S}^*(\beta)$.

Kanas and Wisniowska [4, 5] studied the class of $k-$ uniformly convex functions and the corresponding class of $k-$ uniformly star-like functions related by the Alexander type relation. For $k \geq 0$, the conic domain Ω_k is defined as [6]

$$\Omega_k=\left\{u+iv: u>k\sqrt{(u-1)^2+v^2}\right\}.$$

For a fixed k, Ω_k denotes the conic region that is bounded sequentially by an imaginary axis $(k=0)$, the right branch of hyperbola $(0<k<1)$, a parabola $(k=1)$ and an ellipse $(k>1)$ [7, 8].

Noor et al. [7] define the domain $\Omega_{k,\rho}$ as

$$\Omega_{k,\rho}=(1-\rho)\Omega_{k,\rho}+\rho \quad (0\leq\rho<1).$$

The functions that play the role of extremal functions for these conic regions are specified as

$$h_{k,\rho}(z)=\begin{cases}\frac{1+(1-2\rho)z}{1-z}, & k=0,\\ 1+\frac{2(1-\rho)}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & k=1,\\ 1+\frac{2(1-\rho)}{1-k^2}\sinh^2\left\{\left(\frac{2}{\pi}\arccos k\right)\operatorname{arctan}h\sqrt{z}\right\}, & 0<k<1,\\ 1+\frac{1-\rho}{k^2-1}\sin\left(\frac{\pi}{2K(t)}\int_0^{\frac{\mu(z)}{t}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(xt)^2}}dx\right)+\frac{1-\rho}{k^2-1}, & k>1.\end{cases} \tag{2}$$

A function h such that $h(0)=1$ is known to belong to the class $P(h_{k,\rho})$ if it is subordinate to $h_{k,\rho}(z)$ with $z\in\mathsf{E}$, that is $h(\mathsf{E})\subset h_{k,\rho}(\mathsf{E})=\Omega_{k,\rho}$.

It is not difficult to prove that $P(h_{k,\rho})$ is a convex set. It is known [6] that

$$P(h_k)\subset P\left(\frac{k}{k+1}\right)\subset P, \text{ and also } P(h_{k,\rho})\subset P\left(\frac{k+\rho}{k+1}\right)\subset P.$$

The objective of this paper is to obtain some results for the new classes using the concept of conic domain. We obtained some new results including inclusion results, sufficient condition, Fekete-Szegö inequality and integral preservation properties for these subclasses of multivalent functions. In this sense we give some definitions and notations we need as follows.

Definition 1. *Let* $f\in A(p)$; *then* $f\in k-\mathrm{S}^*(p,\beta)$, *for* β *real and* $|\beta|<\frac{\pi}{2}$, $k\geq 0$, $0\leq\rho<1$, *if and only if*

$$\operatorname{Re}\left\{e^{i\beta}\frac{zf'(z)}{pf(z)}\right\}>k\left|\frac{zf'(z)}{f(z)}-p\right|+\rho\cos\beta \quad (z\in\mathsf{E}). \tag{3}$$

We have the following special cases:

(i) $k-\mathrm{S}^*(p,0)=\mathrm{S}^*_p(k,\rho)$ [9];

(ii) $k-\mathrm{S}^*(1,0)=\mathrm{S}^*(k,\rho)$ [10];

(iii) $0-\mathrm{S}^*(1,\beta)$, the class of $\beta-$ spiral-like functions [11];

(iv) $0-\mathrm{S}^*(1,0)=\mathrm{S}^*(\rho)$ [11,12];

Maejo Int. J. Sci. Technol. **2018**, *12*(02), 139-151

(v) $1-S^*(1,0)=1-S^*\subset S^*(\frac{1}{2})$ [4,5].

Definition 2. *Let* $f\in A(p)$*. Then* $f\in k-C(p,\beta)$*, for* β *real,* $|\beta|<\frac{\pi}{2}$ *and* $k\geq 0$*,* $0\leq\rho<1$ *if and only if*

$$\mathrm{Re}\left\{\frac{e^{i\beta}}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>k\left|\frac{zf''(z)}{f'(z)}-(p-1)\right|+\rho\cos\beta\quad(z\in E). \tag{4}$$

Analogous to the well-known Alexander equivalence [13], we have

$$f\in k-C(p,\beta)\Leftrightarrow\frac{zf'}{p}\in k-S^*(p,\beta).$$

Special cases:
(i) $k-C(p,0)=C_p(k,\rho)$ [9];
(ii) $k-C(1,0)=C(k,\rho)$ [10];
(iii) $0-C(1,\beta)=C^{\beta}(\rho)$ [14];
(iv) $0-C(1,0)=C(\rho)$ [11, 12].

Definition 3. *Let* $f\in A(p)$ *with* $\frac{f'(z)f(z)}{pz}\neq 0$ *for* α*,* β *real,* $|\beta|<\frac{\pi}{2}$*,* $-\frac{1}{2}\leq\gamma<1$*, in* E *and*

$$L(\alpha,\beta,\gamma,f(z))=\left(e^{i\beta}-\alpha\cos\beta\right)\frac{zf'(z)}{pf(z)}+\frac{\alpha\cos\beta}{(p-\gamma)}\left(1-\gamma+\frac{zf''(z)}{f'(z)}\right). \tag{5}$$

Then $f\in k-M(p,\alpha,\beta,\gamma)$ *only if*

$$\mathrm{Re}\,L(\alpha,\beta,\gamma,f(z))>k\left|L(\alpha,\beta,\gamma,f(z))-p\right|+\rho\cos\beta\quad(z\in E).$$

We have the following as special cases:
(i) $0-M(1,\alpha,\beta,0)=SC(\alpha,\beta)$ [15];
(ii) $0-M(1,0,\beta,\gamma)=S^*_p(\beta)$ [11];
(iii) $0-M(1,1,\beta,0)=C(\beta)$ [12];
(iv) $0-M(1,\alpha,0,0)$ [16];
(v) $0-M(1,\alpha,\beta,0)=S^{\beta}_{\alpha}(\rho)$ [17];
(vi) For $\rho=0$, $0-M(1,\alpha,0,0)$, the class of α-star-like functions (of order zero), which has been thoroughly investigated [18, 19, 20].

PRELIMINARY RESULTS

Lemma 1 [21]. *Let* w *be the Schwarz function given by*

$$w(z)=c_1z+c_2z^2+\ldots\qquad(z\in E);$$

then for every complex number μ,

$$\left|c_2-\mu c_1^2\right|\leq 1+\left(|\mu|-1\right)\left|c_1^2\right|.$$

Lemma 2 [22]. *Let* $k\in[0,\infty)$ *be a fixed number and let* h_k *be the function that belongs to the class* $P(h_k)$*. If*

$$h_k(z)=1+P_1(k)z+P_2(k)z^2+\ldots\qquad(z\in E),$$

then

Maejo Int. J. Sci. Technol. **2018**, *12*(02), 139-151

$$P_1 := P_1(k) = \begin{cases} \frac{2A^2}{1-k^2}, & 0 \le k < 1, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\pi^2}{4K^2(t)^2(1+t)\sqrt{t}}, & k > 1, \end{cases} \tag{6}$$

$$P_2 := P_2(k) = D(k)P_1(k), \tag{7}$$

where

$$D(k) = \begin{cases} \frac{A^2+2}{3}, & 0 \le k < 1, \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{(4K(t))^2(t^2+6t+1)-\pi^2}{24K(t)^2(1+t)\sqrt{t}}, & k > 1, \end{cases} \tag{8}$$

$A = \frac{2}{\pi}\arccos k,$ *and* $K(t)$ *is the complete elliptic integral of the first kind* [23].

Lemma 3 [24]. *Let* w *be analytic in* E *with* $w(0) = 0$. *If there exists* $z_0 \in \mathrm{E}$ *such that* $Max_{|z|\le|z_0|} w|z| = w|z_0|$, *then* $z_0 w'(z_0) = m w(z_0)$ *for some* $m \ge 1$.

Lemma 4 [1]. *An analytic function* f *is* $\beta-$ *spiral-like of order* ρ ($0 \le \rho < 1$, $|\beta| < \frac{\pi}{2}$) *if and only if there exists an analytic function* $w(z)$ *satisfying* $w(0) = 0$ *and* $|w(z)| < 1$ *such that*

$$e^{i\beta}\frac{zf'(z)}{pf(z)} = \rho\cos\beta + (1-\rho)(\cos\beta)\frac{1-w(z)}{1+w(z)} + i\sin\beta \quad (z \in \mathrm{E}).$$

MAIN RESULTS

Theorem 1. *Let the function* $h_{k,0}$ *be defined by (2) and* $0 \le k < \infty$ *be a fixed number. If the function f is a member of the function class* $k - \mathrm{M}(p,\alpha,\beta,\gamma)$, *then for* $-\infty < \mu < \infty$,

$$|a_{p+2} - \mu a_{p+1}^2| = \begin{cases} \frac{1}{|\Psi_p(\beta)|}\{\upsilon P_1(k)^2 - P_2(k)\}, & \upsilon > \eta_1(k), \\ \frac{P_1(k)}{|\Psi_p(\beta)|}, & \eta_2(k) \le \upsilon \le \eta_1(k), \\ \frac{1}{|\Psi_p(\beta)|}(P_2(k) - \upsilon P_1(k)^2), & \upsilon < \eta_2(k), \end{cases} \tag{9}$$

where

$$\eta_1(k) = \frac{1+D(k)}{P_1(k)}, \qquad \eta_2(k) = \frac{D(k)-1}{P_1(k)},$$

$$\Psi_p(\beta) = \frac{(e^{i\beta} - \alpha\cos\beta)}{p} + \frac{\alpha\cos\beta}{p^2(1-\gamma)}(2+7p-p^3), \tag{10}$$

$$\upsilon = \left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu - \frac{2+3p+p^2}{2+7p-p^3}\right), \tag{11}$$

and $P_1(k)$, $P_2(k)$ *and* $D(k)$ *are given by (6), (8) and (8) respectively.*

Proof. Let $f \in k - \mathrm{M}(p,\alpha,\beta,\gamma)$; then there exists a Schwarz function w in E such that $w(z) = h_k(w(z))$. A simple computation gives

$$a_{p+1} = \frac{P_1(k)}{\frac{e^{i\beta}-\alpha\cos\beta}{p} + \frac{\alpha\cos\beta}{p(1-\gamma)}(2+p-p^2)}c_1,$$

Maejo Int. J. Sci. Technol. **2018**, *12*(02), 139-151

$$a_{p+2}=\frac{P_1(k)\{c_2+D(k)c_1^2\}}{\frac{2(e^{i\beta}-\alpha\cos\beta)}{p}+\frac{\alpha\cos\beta}{p^2(1-\gamma)}(2+5p-p^3)}+\frac{(p^2+3p+2)P_1(k)^2c_1^2}{(2+7p-p^3)\left\{\frac{(e^{i\beta}-\alpha\cos\beta)}{p}+\frac{\alpha\cos\beta}{p(1-\gamma)}(2+p-p^2)\right\}^2}.$$

Therefore

$$a_{p+2}-\mu a_{p+1}^2=\frac{P_1(k)}{\Psi_p(\beta)}\left[c_2+\left\{D(k)-P_1(k)\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)\right\}c_1^2\right]. \tag{12}$$

Taking modulus on both sides we have

$$\left|a_{p+2}-\mu a_{p+1}^2\right|=\frac{P_1(k)}{\left|\Psi_p(\beta)\right|}\left|c_2-c_1^2+\left\{1+D(k)-P_1(k)\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)\right\}c_1^2\right|.$$

Suppose that $\upsilon>\eta_1(k)$; then using the estimate $\left|c_2-c_1^2\right|\le 1$ from Lemma 1 and the known estimate $|c_1|\le 1$ of the Schwarz Lemma, we have

$$\begin{aligned}\left|a_{p+2}-\mu a_{p+1}^2\right|&\le\frac{P_1(k)}{\left|\Psi_p(\beta)\right|}\left[1+\left\{P_1(k)\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)-D(k)-1\right\}\right]\\&=\frac{P_1(k)^2}{\left|\Psi_p(\beta)\right|}\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)-\frac{P_2(k)}{\left|\Psi_p(\beta)\right|}\\&=\frac{1}{\left|\Psi_p(\beta)\right|}\left\{\upsilon P_1^2(k)-P_2(k)\right\},\end{aligned}$$

where $\Psi_p(\beta)$ and υ are given in (10) and (11)· This is the first inequality of (9).

Now if $\upsilon>\eta_2(k)$; then from (12) we have

$$\left|a_{p+2}-\mu a_{p+1}^2\right|=\frac{P_1(k)}{\left|\Psi_p(\beta)\right|}\left[|c_2|+\left\{D(k)-P_1(k)\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)\right\}|c_1|^2\right].$$

Applying the estimates $|c_2|\le 1-c_1^2$ of Lemma 1 and $|c_1|\le 1$, we have

$$\begin{aligned}\left|a_{p+2}-\mu a_{p+1}^2\right|&\le\frac{P_1(k)}{\left|\Psi_p(\beta)\right|}\left[1+\left\{D(k)-P_1(k)\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)-1|c_1|^2\right\}\right]\\&\le\frac{1}{\left|\Psi_p(\beta)\right|}\left(P_2(k)-\upsilon P_1(k)^2\right).\end{aligned}$$

This is the last expression of (9).

If $\eta_2(k)\le\mu\le\eta_1(k)$, then

$$\left|D(k)-P_1(k)\left(\frac{2+7p-p^3}{3p+p^2-p^3}\right)\left(\mu-\frac{2+3p+p^2}{2+7p-p^3}\right)\right|\le 1.$$

Therefore (12) yields

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\le\frac{P_1(k)}{\left|\Psi_p(\beta)\right|}\left(|c_2|+|c_1|^2\right)\le\frac{P_1(k)}{\left|\Psi_p(\beta)\right|}\left(1-|c_1|^2+|c_1|^2\right)=\frac{P_1(k)}{\left|\Psi_p(\beta)\right|}.$$

Thus, we have the middle inequality of (9).

Maejo Int. J. Sci. Technol. **2018**, *12*(02), 139-151

Theorem 2. *Let* $\alpha > 0$, $|\beta| < \frac{\pi}{2}$; *then*

$$k - \mathrm{M}(p,\alpha,\beta,0) \subset 0 - \mathrm{S}^*(p,\beta).$$

Proof. Let $f \in k - \mathrm{M}(p,\alpha,\beta,0)$ and let

$$w(z) = \frac{e^{i\beta}\frac{zf'(z)}{pf(z)} - cos\beta - isin\beta}{e^{i\beta}\frac{zf'(z)}{pf(z)} - (2\rho - 1)cos\beta - isin\beta},$$

which can be written as

$$e^{i\beta}\frac{zf'(z)}{pf(z)} = \rho\cos\beta + (1-\rho)(\cos\beta)\frac{1-w(z)}{1+w(z)} + i\sin\beta. \tag{13}$$

Clearly $w(0)=0$ in $\mathbb{E}$ and $w(z) \neq -1$. In view of Lemma 4, it is sufficient to show that $|w(z)|<1$. From (13), we have

$$e^{i\beta}\frac{zf'(z)}{pf(z)} = \frac{(2\rho\cos\beta - \cos\beta + i\sin\beta)w(z) + e^{i\beta}}{1+w(z)}. \tag{14}$$

Differentiating (14) logarithmically, with (5) we have

$$\begin{aligned} L(\alpha,\beta,f(z)) = {} & \rho\cos\beta + (1-\rho)(\cos\beta)\frac{1-w(z)}{1+w(z)} + i\sin\beta + \frac{\alpha\cos\beta}{p}\frac{rzw'(z)}{1+rw(z)} \\ & - \frac{\alpha\cos\beta}{p}\frac{zw'(z)}{1+w(z)}, \end{aligned} \tag{15}$$

where

$$r = 2\rho\cos\beta e^{-i\beta} - e^{-2i\beta}.$$

Suppose there exists $\xi \in \mathsf{E}$ such that $\max_{|z|\le|\xi|}|w(z)| = |w(\xi)| = 1$; clearly, $w(\xi) \neq -1$, and if from Lemma 4, there exists $m \geq 1$, $\xi w'(\xi) = mw(\xi)$, we have

$$\begin{aligned} \operatorname{Re} L(\alpha,\beta,f(z)) = \operatorname{Re}\Bigg\{ & \rho\cos\beta - \frac{\alpha m}{2p}\cos\beta + \frac{\alpha m}{2p}\cos\beta \\ & \times\left(\frac{1-2(1-\rho)\cos^2\beta - i2(1-\rho)\sin\beta\cos\beta}{1-(1-\rho)\cos^2\beta - i2\sin\beta\cos\beta}\right)\Bigg\}. \end{aligned}$$

After some simplification we have

$$\begin{aligned} \operatorname{Re} L(\alpha,\beta,f(z)) &= \operatorname{Re}\left\{\rho\cos\beta - \frac{\alpha m}{2p}\cos\beta + \frac{\alpha m}{2p}\cos\beta\frac{l}{u} + i\frac{j}{u}\right\} \\ &= \rho\cos\beta - \frac{\alpha m}{2p}\cos\beta\left(\frac{u-l}{u}\right) < \rho\cos\beta, \end{aligned} \tag{16}$$

where

$$l = 1-(1-\rho)\cos^2\beta\{3-2(1-\rho)\cos^2\beta - 4(1-\rho)\sin^2\beta\}, \tag{17}$$

$$j = \alpha m\cos\beta(1-\rho)^2\sin\beta\cos^3\beta, \tag{18}$$

$$u = \left(1-(1-\rho)\cos^2\beta\right)^2 + \left(2(1-\rho)\sin\beta\cos\beta\right)^2. \tag{19}$$

Consider

Maejo Int. J. Sci. Technol. **2018**, *12*(02), 139-151

$$\begin{aligned} &k\left|L(\alpha,\beta,f(z))-p\right|+\rho\cos\beta \\ &=k\left|\rho\cos\beta-\frac{\alpha m}{2p}\cos\beta+\frac{\alpha m}{2p}\cos\beta\frac{l}{u}+i\frac{j}{u}i\sin\beta-p\right|+\rho\cos\beta \\ &=k\left\{\sqrt{\left(\rho\cos\beta-\frac{\alpha m}{2p}\cos\beta+\frac{\alpha m}{2p}\cos\beta\frac{l}{u}-p\right)^2+\left(\frac{j}{u}+\sin\beta\right)^2}\right\}+\rho\cos\beta \\ &>\rho\cos\beta, \end{aligned} \tag{20}$$

where l, j and u are given by (17), (18) and (19) respectively. From (16) and (20) we have

$$\operatorname{Re} L(\alpha,\beta,f(z))<k\left|L(\alpha,\beta,f(z))-p\right|+\rho\cos\beta.$$

This contradicts the fact that $f\in k-\mathrm{M}(p,\alpha,\beta,0)$. Thus, $|w(z)|<1$ in E. This implies that $f\in 0-\mathrm{S}^*(p,\beta)$. Thus, $k-\mathrm{M}(p,\alpha,\beta,0)\subset 0-\mathrm{S}^*(p,\beta)$.

If we set $p=1,\ \gamma=0,\ k=0$ and $\rho=0,$ we have the following result [15: Theorem 1].

Corollary 1. *If* $f\in \mathrm{SC}(\alpha,\beta)$, *then f is* $\beta-$*spiral-like.*

Corollary 2 [17]. *If* $f\in \mathrm{S}_\alpha^\rho(\beta)$, $\left(\alpha\geq 0, 0\leq\rho<1, |\beta|<\frac{\pi}{2}\right)$, *then f is* $\beta-$*spiral-like of order* ρ.

Theorem 3. *A function f given by (1) belongs to the class* $k-\mathrm{M}(p,\alpha,\beta,0)$ *if and only if there exists a function* $g\in k-\mathrm{S}^*(p,\beta)$ *such that*

$$g(z)=f(z)\left(\frac{zf'(z}{pf(z)}\right)^{\frac{\alpha\cos\beta}{e^{i\beta}}}, \tag{21}$$

where the branch of $\left(\frac{zf'(z}{pf(z)}\right)^{\frac{\alpha\cos\beta}{e^{i\beta}}}$ is chosen to be equal to 1 when $z=0$.

Proof. Let $f\in k-\mathrm{M}(p,\alpha,\beta,0)$; then by Herglotz representation, we have

$$\left(e^{i\beta}-\alpha\cos\beta\right)\frac{zf'(z)}{pf(z)}+\frac{\alpha\cos\beta}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)=\cos\beta\int_0^{2\pi}\left\{1+\left(\frac{1-2\rho ze^{-i\theta}}{1-ze^{-i\theta}}\right)\right\}d\mu(\theta)+i\sin\beta,$$

where $\mu(\theta)$ is a non-decreasing function with $\int_0^{2\pi}d\mu(\theta)=1$.
A simple computation yields

$$\log\frac{f(z)}{z}\left(\frac{z}{f(z)}\right)^{\frac{\alpha\cos\beta}{e^{i\beta}}}\left(f'(z)\right)^{\frac{\alpha\cos\beta}{e^{i\beta}}}=-\frac{\cos\beta}{e^{i\beta}}(1-2\rho)\int_0^{2\pi}\log\left(1-ze^{-i\theta}\right)d\mu(\theta). \tag{22}$$

Now there exist some $g\in k-\mathrm{S}^*(p,\beta)$ such that the right hand side of (22) is equal to $\log\frac{g(z)}{z}$.

It follows that $g(z)=f(z)\left(\frac{zf'(z}{pf(z)}\right)^{\frac{\alpha\cos\beta}{e^{i\beta}}}$.

Conversely, suppose (21) holds true—the logarithmic differentiation of (21) yields

$$\left(e^{i\beta}-\alpha\cos\beta\right)\frac{zf'(z)}{pf(z)}+\frac{\alpha\cos\beta}{p}\left(1+\frac{zf''(z)}{f'(z)}\right)=e^{i\beta}\frac{zg'(z)}{pg(z)}\in \mathrm{P}(h_{k,\rho}).$$

This implies that $f\in k-\mathrm{M}(p,\alpha,\beta,0)$.

If we set $p=1,\ \gamma=0,\ k=0$ and $\rho=0,$ we have the following known result [15: Theorem 2].

Maejo Int. J. Sci. Technol. **2018**, *12*(02), 139-151

Corollary 3. *$f \in \mathrm{SC}(\alpha,\beta)$ if and only if there exists a $\beta-$spiral-like function g such that*

$$g(z) = f(z)\left[\frac{zf'(z)}{f(z)}\right]^{\alpha\cos\beta e^{-i\beta}},$$

where the branch of $\left[\frac{zf'(z)}{f(z)}\right]^{\alpha\cos\beta e^{-i\beta}}$ is chosen to be equal to 1 at $z = 0$.

Theorem 4. *A necessary and sufficient condition for the function f to be in $k-\mathrm{M}(p,\alpha,\beta,0)$, $\alpha \neq 0$, is that f has the integral representation*

$$f(z) = \left[s\int_0^z t^{s-1}\left(\frac{g(t)}{t}\right)^{\frac{e^{i\beta}}{\alpha\cos\beta}} dt\right]^{\frac{1}{s}} \tag{23}$$

for some $g \in k-\mathrm{S}^*(p,\beta)$, where

$$s = 1 + \frac{(e^{i\beta} - \alpha\cos\beta)}{\alpha\cos\beta}. \tag{24}$$

Proof. Suppose $f \in k-\mathrm{M}(p,\alpha,\beta,0)$ and $g \in k-\mathrm{S}^*(p,\beta)$.
From (21), we have

$$\left(f(z)\right)^{\frac{(e^{i\beta}-\alpha\cos\beta)}{\alpha\cos\beta}}\left(f(z)\right)' = \left(\frac{g(z)}{z}\right)^{\frac{e^{i\beta}}{\alpha\cos\beta}} z^{\frac{(e^{i\beta}-\alpha\cos\beta)}{\alpha\cos\beta}}. \tag{25}$$

Integrating (25) from 0 to z , we have (23), where s is given by (24).
Conversely, suppose (23) holds true with $g \in k-\mathrm{S}^*(p,\beta)$ — we have to show that $f \in k-\mathrm{M}(p,\alpha,\beta,0)$.
From (23), after some simplification, we have

$$\left(e^{i\beta} - \alpha\cos\beta\right)\frac{zf'(z)}{pf(z)} + \frac{\alpha\cos\beta}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) = e^{i\beta}\frac{zg'(z)}{pg(z)} \in \mathrm{P}(h_{k,\rho})$$

or

$$\left(e^{i\beta} - \alpha\cos\beta\right)\frac{zf'(z)}{pf(z)} + \frac{\alpha\cos\beta}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \in \mathrm{P}(h_{k,\rho}).$$

This implies that $f \in k-\mathrm{M}(p,\alpha,\beta,0)$.

Theorem 5. *For $0 \le \alpha_1 < \alpha_2$,*

$$k-\mathrm{M}(p,\alpha_2,\beta,\gamma) \subset 0-\mathrm{M}(p,\alpha_1,\beta,\gamma).$$

Proof. Let $f \in k-\mathrm{M}(p,\alpha_2,\beta,\gamma)$.
Now,

$$\frac{1}{p-\gamma}\left[\left(e^{i\beta} - \alpha_1\cos\beta\right)(p-\gamma)\frac{zf'(z)}{pf(z)} + \alpha_1\cos\beta\left(1-\gamma+\frac{zf''(z)}{f'(z)}\right)\right]$$

$$= \frac{\alpha_1}{\alpha_2}\left[\left(e^{i\beta} - \alpha_2\cos\beta\right)\frac{zf'(z)}{pf(z)} + \frac{\alpha_2\cos\beta}{(p-\gamma)}\left(1-\gamma+\frac{zf''(z)}{f'(z)}\right)\right] - \left(\frac{\alpha_1-\alpha_2}{\alpha_2}\right)e^{i\beta}\frac{zf'(z)}{f(z)}$$

$$= \frac{\alpha_1}{\alpha_2}N_1(z) + \left(1-\frac{\alpha_1}{\alpha_2}\right)N_2(z) = N(z),$$

where

Maejo Int. J. Sci. Technol. **2018**, *12*(02), 139-151

$$N_1(z)=\left(e^{i\beta}-\alpha_2\cos\beta\right)\frac{zf'(z)}{pf(z)}+\frac{\alpha_2\cos\beta}{(p-\gamma)}\left(1-\gamma+\frac{zf''(z)}{f'(z)}\right)\in \mathrm{P}(h_{k,\rho})\subset \mathrm{P}(\rho),$$
$$N_2(z)=e^{i\beta}\frac{zf'(z)}{pf(z)}\in \mathrm{P}(\rho)\quad (\text{by Theorem 2}).$$

Since $\mathrm{P}(\rho)$ is a convex set, therefore $N(z)\in \mathrm{P}(\rho)$. This implies that $f\in 0-\mathrm{M}(p,\alpha_1,\beta,\gamma)$. Thus, $k-\mathrm{M}(p,\alpha_2,\beta,\gamma)\subset 0-\mathrm{M}(p,\alpha_1,\beta,\gamma)$.

Theorem 6. *If* $f\in k-\mathrm{S}^*(p,\beta)$, *then* $I_\upsilon(f)\in 0-\mathrm{S}^*(p,\beta)$, *where*

$$I_\upsilon(f)=F(z)=\frac{\upsilon+p}{z^\upsilon}\int_0^z t^{\upsilon-1}f(t)dt;\ \ \upsilon>-1. \tag{26}$$

Proof. Set

$$e^{i\beta}\frac{zF'(z)}{pF(z)}=\rho\cos\beta+(1-\rho)\cos\beta\frac{1-w(z)}{1+w(z)}+i\sin\beta. \tag{27}$$

Clearly, $w(0)=0$ in E. In view of Lemma 4, it is sufficient to show that $|w(z)|<1$.

From (26), we have

$$\frac{zf'(z)}{f(z)}=\frac{e^{-i\beta}\left(2\rho\cos\beta-(1-\upsilon)\cos\beta+i(1+\upsilon)\sin\beta\right)zw'(z)}{e^{-i\beta}\left(2\rho\cos\beta-(1-\upsilon)\cos\beta+i(1+\upsilon)\sin\beta\right)w(z)+1+\upsilon}+\frac{zF'(z)}{F(z)}-\frac{zw'(z)}{1+w(z)}.$$

After some simplification, we have

$$e^{i\beta}\frac{zf'(z)}{pf(z)}=\left\{\rho\cos\beta+(1-\rho)\cos\beta\frac{1-w(z)}{1+w(z)}+i\sin\beta\right.$$
$$\left.+\frac{1}{p}\frac{\left(2\rho\cos\beta-(1-\upsilon)\cos\beta+i(1+\upsilon)\sin\beta\right)zw'(z)}{\left((\rho-1)\cos^2\beta-i(\rho-1)\sin\beta\cos\beta\right)w(z)+1+\upsilon}-\frac{e^{i\beta}}{p}\frac{zw'(z)}{1+w(z)}\right\}.$$

Suppose there exists $\xi\in\mathsf{E}$ such that $\max_{|z|\le|\xi|}|w(z)|=|w(\xi)|=1$; clearly, $w(\xi)\neq-1$, and if from Lemma 3, there exists $m\ge 1$; $\xi w'(\xi)=mw(\xi)$, we have

$$\begin{aligned}\operatorname{Re}e^{i\beta}\frac{zf'(z)}{f(z)}&=\operatorname{Re}\left(\rho\cos\beta-\frac{m}{2p}\cos\beta+\frac{m}{2p}\frac{\Lambda}{\Delta}+i\frac{m}{2p}\frac{\Gamma}{\Delta}\right)\\&=\rho\cos\beta-\frac{m}{2p}\left(\frac{\Delta\cos\beta-\Lambda}{\Delta}\right)<\rho\cos\beta,\end{aligned} \tag{28}$$

where

$$\Lambda=\left(2\rho+\upsilon-1\right)\left((\upsilon+1)+(\rho-1)\cos^2\beta\right)\cos\beta, \tag{29}$$
$$\Delta=\left((\upsilon+1)+(\rho-1)\cos^2\beta\right)^2+\left((\rho-1)\sin\beta\cos\beta\right)^2, \tag{30}$$
$$\Gamma=\left\{2(\rho+\upsilon)(\rho-1)\cos^2\beta+(1+\upsilon)^2\right\}\sin\beta. \tag{31}$$

Consider

$$k\left|\frac{zf'(z)}{f(z)}-p\right|+\rho\cos\beta$$
$$=k\left|\rho\cos\beta-\frac{m}{2p}\cos\beta+\frac{m}{2p}\frac{\Lambda}{\Delta}-i\frac{m\sin\beta}{2}+i\sin\beta+i\frac{m}{2p}\frac{\Gamma}{\Delta}-p\right|+\rho\cos\beta$$

This can be written as

Maejo Int. J. Sci. Technol. **2018**, *12*(02), 139-151

$$
\begin{aligned}
&k\left|\frac{zf'(z)}{f(z)}-p\right|+\rho\cos\beta\\
&=k\left\{\sqrt{\left(\rho\cos\beta-\frac{m}{2p}\cos\beta+\frac{m}{2p}\frac{\Lambda}{\Delta}-p\right)^2+\left(\sin\beta-\frac{m\sin\beta}{2}+\frac{m}{2p}\frac{\Gamma}{\Delta}\right)^2}\right\}+\rho\cos\beta \qquad (32)\\
&>\rho\cos\beta,
\end{aligned}
$$

where Λ, Δ and Γ are given by (29), (30) and (31) respectively. From (28) and (32) we have

$$\operatorname{Re}\left\{e^{i\beta}\frac{zf'(z)}{pf(z)}\right\}<k\left|\frac{zf'(z)}{f(z)}-p\right|+\rho\cos\beta.$$

This contradicts the fact that $f\in k-\mathrm{S}^*(p,\beta)$. Thus, $|w(z)|<1$ in E and $F\in 0-\mathrm{S}^*(p,\beta)$ $(z\in\mathsf{E})$.

Theorem 7. *A function $f\in A(p)$ satisfies the condition*

$$\left|\frac{1}{e^{i\beta}F(z)}-\frac{1}{2\rho}\right|<\frac{1}{2\rho} \qquad (33)$$

if and only if $f\in 0-\mathrm{S}^*(p,\beta)$, where $F(z)=\frac{zf'(z)}{pf(z)}$.

Proof. Suppose f satisfies (33) —then we can write

$$
\begin{aligned}
&\left|\frac{2\rho-e^{i\beta}F(z)}{2\rho e^{i\beta}F(z)}\right|<\frac{1}{2\rho}\Leftrightarrow\left|\frac{2\rho-e^{i\beta}F(z)}{2\rho e^{i\beta}F(z)}\right|^2<\left(\frac{1}{2\rho}\right)^2\\
&\Leftrightarrow\left(2\rho-e^{i\beta}F(z)\right)\left(\overline{2\rho-e^{i\beta}F(z)}\right)<\left(e^{-i\beta}\overline{F(z)}\right)e^{i\beta}F(z)\\
&\Leftrightarrow 4\rho^2-2\rho e^{-i\beta}\overline{F(z)}-2\rho e^{i\beta}F(z)+F(z)\overline{F(z)}<F(z)\overline{F(z)}\\
&\Leftrightarrow 4\rho^2-2\rho\left(e^{-i\beta}\overline{F(z)}+e^{i\beta}F(z)\right)<0\\
&\Leftrightarrow \operatorname{Re}\left(e^{i\beta}F(z)\right)>\rho.
\end{aligned}
$$

This completes the proof.

Theorem 8. *Let $\alpha\geq 1$ and $|\beta|<\frac{\pi}{2}$; then*

$$k-\mathrm{M}(p,\alpha,\beta,0)\subset\mathsf{C}(p,\beta).$$

Proof. Let $f\in k-\mathrm{M}(p,\alpha,\beta,0)$; then from (19) we have

$$f(z)=\left[s\int_0^z t^{s-1}\left(\frac{g(t)}{t}\right)^{\frac{e^{i\beta}}{\alpha\cos\beta}}dt\right]^{\frac{1}{s}},$$

where s is given by (24).

Differentiating the above equation, we obtain

$$zf'(z)\left(f(z)\right)^{s-1}=\left(g(z)\right)^{\frac{e^{i\beta}}{\alpha\cos\beta}}. \qquad (34)$$

Logarithmic differentiation of (34) yields

$$\frac{\left(zf'(z)\right)'}{f'(z)}=(1-s)\frac{zf'(z)}{f(z)}+\frac{e^{i\beta}}{\alpha\cos\beta}\frac{zg'(z)}{g(z)},$$

or equivalently, this can be written as

$$\frac{\cos\beta}{p}\left(\frac{zf'(z)}{f'(z)}\right)' + i\sin\beta\,\frac{zf'(z)}{pf'(z)} = \left(1-\frac{1}{\alpha}\right)e^{i\beta}\,\frac{zf'(z)}{pf'(z)} + \frac{1}{\alpha}e^{i\beta}\,\frac{zg'(z)}{pg(z)}$$
$$= \left(1-\frac{1}{\alpha}\right)R_1(z) + \frac{1}{\alpha}R_2(z) = R(z),$$

where

$$R_1(z) = e^{i\beta}\,\frac{zf'(z)}{pf(z)} \in \mathrm{P}(\rho) \quad \text{(by Theorem 2)},$$

$$R_2(z) = e^{i\beta}\,\frac{zg'(z)}{pg(z)} \in \mathrm{P}(\rho) \qquad \text{(by hypothesis)}.$$

Since $\mathrm{P}(\rho)$ is a convex set, therefore $R(z) \in \mathrm{P}(\rho)$. This completes the proof.

Theorem 9. *If* $f \in A(p)$ *and satisfies*

$$\sum_{n=1}^{\infty}\left\{\frac{n}{p} + \left|\frac{n}{p}e^{i\beta} + 2(1-\rho)\cos\beta\right|\right\}\left|a_{n+p}\right| < 2(\rho-1)\cos\beta, \tag{35}$$

then $f \in k-\mathrm{S}^*(p,\beta)$.

Proof. We assume that the inequality (35) holds true. It suffices to show that

$$\left|\frac{e^{i\beta}\left(\frac{zf'(z)}{pf(z)}-1\right)}{e^{i\beta}\,\frac{zf'(z)}{pf(z)} - \{(2\rho-1)\cos\beta + i\sin\beta\}}\right| < 1 \quad (z \in \mathrm{E}).$$

We have

$$\left|\frac{e^{i\beta}\left(\frac{zf'(z)}{pf(z)}-1\right)}{e^{i\beta}\,\frac{zf'(z)}{pf(z)} - \{(2\rho-1)\cos\beta + i\sin\beta\}}\right| = \left|\frac{e^{i\beta}\sum_{n=1}^{\infty}\frac{n}{p}a_{n+p}z^{n+p}}{\{2(1-\rho)\cos\beta\}z^p + \sum_{n=1}^{\infty}\left(\frac{n}{p}e^{i\beta} + 2(1-\rho)\cos\beta\right)a_{n+p}z^{n+p}}\right|$$
$$= \frac{\sum_{n=1}^{\infty}\frac{n}{p}\left|a_{n+p}\right|}{\left|2(\rho-1)\cos\beta\right| - \sum_{n=1}^{\infty}\left|\frac{n}{p}e^{i\beta} + 2(1-\rho)\cos\beta\right|\left|a_{n+p}\right|}.$$

The last expression is bounded above by 1 if

$$\sum_{n=1}^{\infty}\frac{n}{p}\left|a_{n+p}\right| < 2(\rho-1)\cos\beta - \sum_{n=1}^{\infty}\left|\frac{n}{p}e^{i\beta} + 2(1-\rho)\cos\beta\right|\left|a_{n+p}\right|,$$

which is equivalent to the inequality (35). Hence we have $f \in k-\mathrm{S}^*(p,\beta)$.

If we set $p=1$, $k=0$ and $\beta=0$, we have the following result.

Corollary 4. *If* $f \in A$ *satisfies*

$$\sum_{n=1}^{\infty}\{n+1-\rho\}\left|a_{n+1}\right| < \rho-1 \qquad (z \in \mathrm{E}),$$

then $f \in \mathrm{S}^*(\rho)$.

Corollary 5. *If* $f \in A(p)$ *and satisfies*

$$\sum_{n=1}^{\infty}\left\{n+p-1+\left|(n+p+1)e^{i\beta} - 2\rho\cos\beta\right|\right\}\left|a_{n+p}\right| < 2p\{(\rho-1)\cos\beta - (p-1)\},$$

then $f \in k-\mathrm{C}(p,\beta)$.

Corollary 6. *If a function* $f \in A$ *satisfies*

$$\sum_{n=1}^{\infty}(n+1-2\rho)<\rho-1 \qquad (z\in \mathrm{E}),$$

then $f\in \mathrm{C}(\rho)$.

CONCLUSIONS

In this paper we have used the concept of spiral-like functions and generalised conic domain to introduce some new classes of analytic multivalent functions in an open unit disk. The main results are inclusion relations, integral preserving property, Fekete-Szego inequality, sufficient condition, integral representation theorem and some other results. We have also deduced some known results from our main results.

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