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Full Paper

On left permutable inverse LA-semigroups

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Abstract: We investigate some fundamental properties of left permutable inverse LA-semigroups in which inverses commute. We prescribe a natural partial order and order ideal on an inverse LA-semigroup. Then we use the partial order to show that the set of idempotent elements in a left permutable inverse LA-semigroup is an order ideal. Also, we introduce the compatibility relations to ascertain important characteristics of meets and joins in the left permutable inverse LA-semigroups. Finally, we establish the conditions under which a left permutable inverse LA-semigroup is infinitely distributive.

Keywords: **i**nverse LA-semigroup, partial order, order ideal, infinitely distributive, compatibility relation

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INTRODUCTION

A left almost semigroup (LA-semigroup) is defined as a groupoid which satisfies the left invertive law $(ab)c = (cb)a$. Similarly, the right almost semigroup is a groupoid which satisfies the right invertive law $a(bc) = c(ba)$. These notions were introduced by Kazim and Naseeruddin in 1970s [1, 2]. Later, several fundamental properties and new notions related to LA-semigroups were discussed in the literature [3-8]. An LA-semigroup is also known as a right modular groupoid, a left invertive groupoid or an Abel Grassmann's groupoid (AG-groupoid) [9-11]. The name AG law was used for the first time by Denes and Keedwell [12] to represent the identity $a(bc) = b(ac)$. Later, Protić and Stevanović [11] used the name AG-groupoid for an LA-semigroup because in every LA-semigroup the medial law holds naturally. An LA-semigroup is a non-associative structure midway between a groupoid and a commutative semigroup. For example, an LA-semigroup with right identity is a commutative semigroup.

By the successive application of the left invertive law in an arbitrary LA-semigroup, the medial law holds naturally, i.e. $(ab)(cd) = (ac)(bd)$. It is important to mention here that every

LA-semigroup is a medial groupoid, but its converse is not true. For instance, $(0 * 3) * 1 \neq (1 * 3) * 0$ in Table 1, which shows that $\boldsymbol{A} = \{0,1,2,3,4\}$ is not an LA-semigroup, whereas $\boldsymbol{A}$ is medial. Therefore, we continue calling it an LA-semigroup rather than an AG-groupoid.

Table 1. A medial groupoid which does not satisfy left invertive law

*	0	1	2	3	4
0	4	4	4	2	4
1	4	0	0	4	4
2	4	0	4	2	4
3	0	2	2	0	0
4	4	4	4	4	4

An inverse LA-semigroup is an LA-semigroup $\boldsymbol{L}$, in which for every $a \in \boldsymbol{L}$ there exists $a' \in \boldsymbol{L}$ such that $(aa')a = a$ and $(a'a)a' = a'$. Let $\boldsymbol{\mathcal{I}}$ ={**€:** € is an inverse LA-semigroup} represent the class containing all inverse LA-semigroups. The concept of € was given by Mushtaq and Iqbal [13]; they investigated some fundamental properties of inverses in € satisfying weak associative law, i.e. $(ab)c = b(ac)$. One can see that in € with weak associative law the identities $(ab)c = b(ac)$ and $(ab)c = b(ca)$ are equivalent. Therefore, the identity $a(bc) = a(cb)$ also holds in €. Furthermore,

$$ab = ((aa')a)b = a((aa')b) = a((ba')a) = (aa)(ba') = (aa)(a'b) = a((a'b)a) =$$
$$a((ab)a') = (a'a)(ab) = (a'a)(ba) = (a'b)(aa) = b(a'(aa)) = b((aa')a) = ba$$

imply that € is commutative. It follows therefore that $(ab)c = (ba)c = a(bc)$ for every $a, b, c \in$ €. Consequently, € becomes a commutative inverse semigroup.

Protić and Božinović [14] defined that an LA-semigroup satisfying the left permutable law $a(bc) = b(ac)$ is called LA**-semigroup. They also proved that an LA-semigroup with left identity is LA**-semigroup and every LA**-semigroup satisfies the paramedial law $(st)(uv) = (vt)(us)$.

A left permutable inverse LA-semigroup is a groupoid satisfying the identities $(ab)c = (cb)a$, $a(bc) = b(ac)$ and contains inverse a' for each element a, i.e. $(aa')a = a$ and $(a'a)a' = a'$. Alternatively, a left permutable inverse LA-semigroup is an € $\in \boldsymbol{\mathcal{I}}$ satisfying the identity $a(bc) = b(ac)$. An example of the left permutable inverse LA-semigroup, depicting that it may not contain an identity element is given in Table 2.

Table 2. A left permutable inverse LA-semigroup without any left identity

*	0	1	2	3	4
0	1	1	3	3	1
1	1	1	1	1	1
2	0	1	1	1	4
3	0	1	1	1	0
4	1	1	2	3	1

Let $\boldsymbol{\mathcal{L}}$ ={**£:** £ is left permutable inverse LA-semigroup} represent the class containing all left permutable inverse LA-semigroups. Božinović et al. [15] provided the concept of a natural partial order in £ to introduce the notions of a pseudo normal congruence pair, a normal congruence pair and kernel normal system of £.

Here, we investigate £ in which inverses commute, i.e. $aa' = a'a$ for all $a \in$ £. Let $\boldsymbol{E}(£)$ be a set of idempotent elements in £. It is already proved that in £, $aa' = a'a$ if and only if $aa',\ a'a \in \boldsymbol{E}(£)$ [15]. We produce some significant results in £ which will be used further in this paper. We prove a few interesting properties based on the natural partial order in £. We also define a compatibility relation in £ and use this to prove certain results of meets and joins. We discuss homomorphism between two inverse LA-semigroups. Also, we furnish the conditions under which £ is infinitely distributive. The notions discussed in this study are related to the study of inverse semigroups [16-17].

The purpose of this article is to look for connections between natural partial orders and compatibility relations. The motivation behind this research is that such partial orders give insight into LA-semigroups and their behaviours similar to commutative semigroups despite their non-commutative and non-associative algebraic structure. We also study the complementary behaviour of meets and joins with respect to the natural partial order.

NATURAL PARTIAL ORDER AND COMPATIBILITY RELATION

Mushtaq and Iqbal [13] investigated some basic characteristics of inverses in an inverse LA-semigroup. We establish some standard results regarding the natural partial orders by using the fundamental properties of an inverse LA-semigroup discussed in Proposition 1.

Proposition 1. Let £ $\in \boldsymbol{\mathcal{L}}$. Then

i) $(a')' = a$ for all $a \in$ £;

ii) If e is an idempotent element of £, then $e' = e$;

iii) $\Big(\big(\big((a_1a_2)a_3\big)\dots\big)a_n\Big)' = \Big(\big(\big((a_1'a_2')a_3'\big)\dots\big)a_n'\Big)$ for all $a_1, a_2, \dots, a_n \in$ £, $n \geq 2$.

Proof. i) Obviously, a is the solution of equations $a' = (a'b)a'$ and $b = (ba')b$. Consequently, by uniqueness of inverses, $(a')' = a$.

ii) It is immediate from the definition.

iii) For $n = 2$, it is clear by definition that $\big((a_1a_2)(a_1'a_2')\big)(a_1a_2) = \big((a_1a_1')(a_2a_2')\big)(a_1a_2) = \big((a_1a_1')a_1\big)\big((a_2a_2')a_2\big) = a_1a_2$ and $\big((a_1'a_2')(a_1a_2)\big)(a_1'a_2') = a_1'a_2'$, i.e. $(a_1a_2)' = a_1'a_2'$. By using induction, it is now straight forward to generalise the result. □

In an LA-semigroup $\boldsymbol{L}$ with left identity e, an element $u^{-1} \in \boldsymbol{L}$ is called a left (right) inverse of $u \in \boldsymbol{L}$ if $u^{-1}u = e$ ($uu^{-1} = e$). Also, if u^{-1} is a left inverse of u, then $uu^{-1} = (eu)u^{-1} = (u^{-1}u)e = ee = e$. Consequently, any left inverse is also the right inverse in $\boldsymbol{L}$ and so is the inverse. In particular, if $v \in \boldsymbol{L}$ is another left inverse of u, then $u^{-1} = eu^{-1} = (vu)u^{-1} = (u^{-1}u)v = ev = v$. This means that the left inverse of each element in $\boldsymbol{L}$ is unique. Furthermore, if f is another left identity of $\boldsymbol{L}$, then $f = ef = (ee)f = (fe)e = ee = e$. This implies that e is the unique left identity of $\boldsymbol{L}$.

An LA-semigroup containing a left identity is called an LA-monoid. Additionally, an LA-monoid containing a unique left inverse for each of its element is called an LA-group. Every LA-group contains one idempotent element only, which is its left identity.

The following proposition provides a connection between a left permutable inverse LA-semigroup and an LA-group.

Proposition 2. Every £ $\in \boldsymbol{\mathcal{L}}$, in which $aa' = a'a$ for all $a \in$ £, with a unique idempotent element is precisely an LA-group.

Proof. Since every LA-group contains a left identity, therefore it is a left permutable inverse LA-semigroup with one idempotent only. Conversely, suppose £ $\in \boldsymbol{\mathcal{L}}$ has only one idempotent e. Then $a'a = e = aa'$ for every $a \in$ £. But $ea = (aa')a = a$. Therefore, e is the left identity of **£.** Hence, £ is an LA-group. □

The relation $\leq$ on any £ $\in \boldsymbol{\mathcal{L}}$ is defined as follows: $a \leq b$ if and only if $a = eb$ for some idempotent *e*. Let Q be a subset of partial order set P. If $b \leq c \in Q$ implies $b \in Q$, then Q is called an order ideal. Moreover, $[b] = \{c \in P: c \leq b\}$ is called the principal or least-order ideal produced by b. Alternatively, $[Q] = \{c \in P: c \leq a$ for some $a \in Q\}$ is an order ideal which is generated by a subset Q of P.

Before proving that the relation ' $\leq$ ' defined above is a partial order, we use this concept to prove the following lemma, which is a basis of the forthcoming results.

Lemma 1. Let £ $\in \boldsymbol{\mathcal{L}}$ and $aa' = a'a$ for all $a \in$ £. Then for all $a, b \in$ £, the following are equivalent:

i) $a \leq b$;
ii) $a' \leq b'$;
iii) $a = (aa')b$.

Proof. i) ⇒ ii): Let $a \leq b$. Then $a = eb$ for some $e \in \boldsymbol{E}($£$)$. So $a' = eb'$ by Proposition 1. Hence $a' \leq b'$.
ii) ⇒ iii): Let $a' \leq b'$. Then $a' = eb'$ for some $e \in \boldsymbol{E}($£$)$, which implies that $a = eb$. Moreover, $a'a = (ea')a = (aa')e = e(aa')$. Thus, $a = (aa')b$.
iii) ⇒ i): Let $a = (aa')b$. Since aa' is an idempotent, then by definition $a \leq b$. □

In (ii) of the above lemma, it is important to point out that here the inversion is not reversing the relation as in group theory or other algebras.

Proposition 3. Let £ $\in \boldsymbol{\mathcal{L}}$ and $aa' = a'a$ for all $a \in$ £. Then

i) $\leq$ is a partial order relation;
ii) $e_1 \leq e_2$ if and only if $e_1 = e_2e_1 = e_1e_2$ for all $e_1, e_2 \in \boldsymbol{E}($£$)$;
iii) If $a \leq b$ and $x \leq y$, then $xa \leq yb$;
iv) If $a \leq b$, then $aa' \leq bb'$ and $a'a \leq b'b$;
v) The set of idempotent elements of £ is an order ideal.

Proof. i) Since $x = (xx')x$ implies that $x \leq x$, the relation $\leq$ is reflexive. Let $x \leq$ y and y $\leq x$. Then $x = (xx')y$ and $y = (yy')x$, so that $x = (xx')y = (xx')((yy')x) = (yy')((xx')x) = (yy')x = y$. Therefore, the relation $\leq$ is antisymmetric. Suppose that $x \leq y$ and $y \leq z$. Then $x = (xx')$ and $y = (yy')z$. Hence, $x = (xx')y = (xx')((yy')z) = (zx')((yy')x) = (z(yy'))(x'x) = ((x'x)(yy'))z$, i.e. $x \leq z$.
ii) Let $e_1 \leq e_2$, then $e_1 = i\, e_2$ for some $i \in \boldsymbol{E}($£$)$. But $e_1e_2 = e_1$. Now $e_2e_1 = (e_2e_2)e_1 = (e_1e_2)e_2 = e_1e_2 = e_1$. The converse follows obviously.
iii) If $a \leq b$ and $x \leq y$, then $a = (aa')b$ and $x = (xx')y$. Now $ax = ((aa')b)((xx')y) = ((aa')(xx'))by$, which implies that $ax \leq by$.
iv) It is obvious, because of Lemma 1 and Proposition 3(iii).
v) Evidently, $\boldsymbol{E}($£$) \subseteq$ £. For $a \leq b \in \boldsymbol{E}($£$)$, $a = (aa')b$ by definition. Which immediately means

□

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that $a \in \boldsymbol{E}(£)$. Therefore $\boldsymbol{E}(£)$ is an order ideal of £.

In the preceding proposition, the defined relation $\leq$ is a partial order. This partial order is known as natural partial order because such an order can provide more knowledge about an inverse LA-semigroup in a specific way since it follows the binary operation in a special sense. It is interesting to note that this natural partial order is compatible with respect to the multiplication defined in £. Moreover, the natural partial order defines an order ideal on the set of idempotents of £.

The left compatibility relation on any € is defined by $b\sim_l c$ if and only if $bc' \in \boldsymbol{E}(€)$, whereas the right compatibility relation is defined by $b\sim_r c$ if and only if $b'c \in \boldsymbol{E}(€)$ for all $b, c \in$ €. The compatibility relation is a relation which is both left and right compatibility relation. These relations are reflexive and symmetric, but they are not transitive. However, from our investigation, the left compatibility relation and the right compatibility relation coincide in any €. Because $b\sim_l c$ if and only if $bc' \in E(€)$, therefore $(b'c)^2 = ((bc')')^2 = ((bc')^2)' = (bc')' = bc'$. This implies that $b\sim_r c$. So the left compatibility relation or right compatibility relation is the compatibility relation in €.

Let $(P, \leq)$ be a partial order set. If $c \leq a, b$, then c is known as a lower bound. If c is the biggest lower bound among all the pairs a and b, then c is said to be the greatest lower bound and is written as $a \wedge b$. A meet-semilattice is a partial order set containing the greatest lower bound for every pair of elements.

Now we prove some important conditions which relate a compatibility relation to a natural partial order defined on any £.

Lemma 2. Let £ $\in \boldsymbol{\mathcal{L}}$ and $aa' = a'a$ for all $a \in$ £. If b~c, then the greatest lower bound $b \wedge c$ of b and c exists and $(b \wedge c)'(b \wedge c) = ((b'b)c')c$.

Proof. If $b\sim c$, then by definition $bc', b'c \in \boldsymbol{E}(£)$. Let $z = (bc')c = (cc')b$. Then $z \leq c$ and $z \leq b$. If $w \leq b$ and $w \leq c$, then $ww' \leq bc'$ and so $w = (ww')w \leq (bc')c = z$ by Proposition 3(iii). Hence $z = b \wedge c$. Moreover, $z'z = ((bc')c)'((bc')c) = ((bc')'c')((bc')c) = ((b'c)c')((bc')c)$ $= ((b'c)(bc'))(c'c) = ((b'b)(cc'))(c'c) = ((b'b)c')((cc')c) = ((b'b)c')c$. □

Lemma 3. Let £ $\in \boldsymbol{\mathcal{L}}$ and $aa' = a'a$ for all $a \in$ £. If b~c, then

$$b \wedge c = (bc')c = (b'c)c = (cc')b = (cc)b' = (cb')b = (c'b)b.$$

Proof. It is immediate by the fact that $b \wedge c = (bc')c$. Since bc' is an idempotent, so obviously, $bc' = (bc')' = b'c$. □

Lemma 4. Let £ $\in \boldsymbol{\mathcal{L}}$ and $aa' = a'a$ for all $a \in$ £. Then

i) $b\sim c$ and $u\sim v$ imply $bu\sim cv$;
ii) $b \leq c$, $u \leq v$ and $c\sim v$ imply $b\sim u$.

Proof. i) If $b\sim c$ and $u\sim v$, then $bc', uv' \in \boldsymbol{E}(£)$. Also, $(bu)(cv)' = (bu)(c'v') = (bc')(uv') \in \boldsymbol{E}(£)$; therefore $bu\sim cv$.
ii) Let $b \leq c$, $u \leq v$ and $c\sim v$. Then $b = (bb')c,\ u = (uu')v$ and $cv' \in \boldsymbol{E}(£)$. This implies that $bu' = ((bb')c)((uu')v)' = ((bb')c)((u'u)v') = ((bb')(u'u))(cv') \in \boldsymbol{E}(£)$. Hence $b\sim u$. □

Any subset $\boldsymbol{B}$ of € is called compatible if every pair of B is compatible.

Lemma 5. For elements a, b and c of £ $\in \boldsymbol{\mathcal{L}}$ and $a'a = aa'$ for all $a \in$ £,

i) $b\sim c$ and $b'b \leq c'c$ imply $b \leq c$.

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ii) $[a]$ is a compatible subset of £.

Proof. i) It is obvious.
ii) Let $b, c \in [a] = \{b \in P: b \leq a\}$. Then by definition, $b \leq a$ and $c \leq a$, which implies that $b = ea$ and $c = fa$. Hence $bc' = (ea)(fa)' = (ea)(fa') = (ef)(aa') \in \boldsymbol{E}(£)$. Consequently, $[a]$ is a compatible subset of £. □

HOMOMORPHISMS BETWEEN INVERSE LA-SEMIGROUPS

The concept of homomorphisms between inverse LA-semigroups is the same as in semigroups. For example, a mapping $\varphi: £ \to £$ defined by $\varphi(a) = a'$ for all $a \in £$. Then φ is a homomorphism by virtue of Proposition 1. A mapping $\phi: P_1 \to P_2$ between two partial order sets P_1 and P_2 is order preserving if $a \leq b$ means $\phi(a) \leq \phi(b)$. It is important to mention here that if an order preserving ϕ is bijective and its inverse also preserves order, then ϕ is called an order isomorphism.

Proposition 4. Let $£_1$, $£_2 \in \boldsymbol{\mathcal{L}}$ and $\phi: £_1 \to £_2$ be a homomorphism. Then

i) $\phi(a') = \phi(a)'$ for all $a \in £_1$;
ii) if $e \in \boldsymbol{E}(£_1)$, then $\phi(e) \in \boldsymbol{E}(£_2)$;
iii) if $\phi(a) \in \boldsymbol{E}(£_2)$, then there exists $e \in \boldsymbol{E}(£_1)$ such that $\phi(a) = \phi(e)$;
iv) the function ϕ is order preserving;
v) if $b, c \in £_1$ such that $\phi(b) \leq \phi(c)$, then there must be an element $d \in £_1$ for which $d \leq c$ and $\phi(d) = \phi(a)$.

Proof. i) Since $(\phi(a)\phi(a'))\phi(a) = \phi(aa')\phi(a) = \phi((aa')a) = \phi(a)$ and $(\phi(a')\phi(a))\phi(a') = \phi(a'a)\phi(a') = \phi((a'a)a') = \phi(a')$, therefore due to the uniqueness of inverses, $\phi(a') = \phi(a)'$.
ii) It is immediate from the fact that $\phi(e)^2 = \phi(e)\phi(e) = \phi(ee) = \phi(e)$.
iii) If $\phi(a)^2 = \phi(a)$, then $\phi(a'a) = \phi(a')\phi(a) = \phi(a)'\phi(a) = \phi(a)\phi(a) = \phi(a)^2 = \phi(a)$. Since $a'a \in \boldsymbol{E}(£_1)$, take $a'a = e$; then $\phi(a) = \phi(e)$.
iv) Let $a_1 \leq a_2$. Then $a_1 = ea_2$ for some $e \in \boldsymbol{E}(£_1)$, which implies that $\phi(a_1) = \phi(ea_2) = \phi(e)\phi(a_2)$ and $\phi(e)$ is an idempotent. Thus, $\phi(a_1) \leq \phi(a_2)$.
v) Take $d = (bb')c$. Then $d \leq c$ and $\phi(d) = \phi(bb')\phi(c) = \phi(b)$. □

We use a single word subset instead of non-empty subset throughout this section.

Lemma 6. Let $€ \in \boldsymbol{\mathcal{I}}$ and B be a set of idempotent elements. Then

i) $\wedge B$ is an idempotent if it exists;
ii) $\vee B$ is an idempotent if it exists.

Proof. i) It is straightforward because of the fact that the set of idempotent elements is an order ideal.
ii) Let $\vee B = b$. Then $e \leq b$ for all $e \in B$, which shows that $e \leq b'b$ by Proposition 3(iv). Consequently, $b \leq b'b$ because b is also an idempotent. □

Lemma 7. Let $£ \in \boldsymbol{\mathcal{L}}$ and $aa' = a'a$ for all $a \in £$, and let $\{b_i: i \in I\}$ be a subset of £ and $b = \vee B$ exist. Then $c \sim d$ for all $c, d \in B$.

Proof. Let $c, d \in B$. Then by definition $c, d \leq b$. Now by Lemma 4(ii), $a \sim b$.□

Proposition 5. Let $£ \in \boldsymbol{\mathcal{L}}$ and $aa' = a'a$ for all $a \in £$, and let $\{b_i: i \in I\}$ be a subset of £.

i) If $\vee b_i$ exists, then $\vee b_i b_i'$ exists and $(\vee b_i)(\vee b_i)' = \vee b_i b_i'$.

ii) If $\vee b_i$ exists, then $\vee b_i' b_i$ exists and $(\vee b_i)'(\vee b_i) = \vee b_i' b_i$.

Proof. i) If $b = \vee b_i$, then $b_i \le b$ implies that $b_i b_i' \le bb'$. Furthermore, bb' is an upper bound of $\{b_i b_i' : i \in I\}$. Suppose that $b_i b_i' \le c$ for some $c \in £$ and for all $i \in I$. Then $b_i = (b_i b_i')b_i \le cb_i \le cb$ for all $i \in I$. Certainly, $b = \vee b_i \le cb$. Furthermore, $b = (bb')(cb) = c((bb')b) = cb$ by Lemma 1, which directly means that $bb' = (cb)b' = (b'b)c$. It follows that $bb' \le c$. Hence $\vee b_i b_i' = bb'$.

ii) It is obvious due to (i). □

Proposition 6. Let $£ \in \mathcal{L}$ and $aa' = a'a$ for all $a \in £$, and let $\{b_i : i \in I\}$ be a subset of £.

i) If $b = \vee b_i$ and $b_i b_i' \le a'a$ for all $i \in I$, then $\vee b_i a$ exists and $ba = \vee b_i a$.

ii) If $b = \vee b_i$ and $b_i' b_i \le aa'$ for all $i \in I$, then $\vee ab_i$ exists and $ab = \vee ab_i$.

iii) If $b = \wedge b_i$ exists, then $\wedge b_i a$ exists and $\wedge b_i a = ba$.

iv) If $b = \wedge b_i$ exists, then $\wedge ab_i$ exists and $\wedge ab_i = ab$.

Proof. i) Let $b = \vee b_i$. Then $b_i \le b$ for all $i \in I$, which implies that $b_i a \le ba$ for each $i \in I$. It follows that ba is an upper bound of $\{b_i a : i \in I\}$. If $b_i a \le c$ for some $c \in £$ and for each $i \in I$, then $(b_i a)a' \le ca'$ implies that $(a'a)b_i \le ca'$. Since $b_i b_i' \le a'a$, therefore $(b_i b_i')b_i \le (a'a)b_i \le ca'$. Hence $b_i \le ca'$. It is then immediate that $ba \le (ca')a = (aa')c \le c$ and $ba = \vee b_i a$.

ii) It is obvious from (i).

iii) Since $b \le b_i$ for each $i \in I$, therefore $ba \le b_i a$ for each $i \in I$, and so ba is an upper bound of $\{b_i a : i \in I\}$. If $c \le b_i a$ for some $c \in £$ and for each $i \in I$, then $ca' \le (b_i a)a' = (a'a)b_i \le b_i$ shows that $ca' \le \wedge b_i = b$. Moreover, $(cc')c \le ((b_i a)(b_i' a'))c = ((b_i b_i')(aa'))c \le (aa')c = (ca')a \le ba$. Which is immediate that $c \le ba$ and $\wedge b_i a = ba$.

iv) It is similar to (iii). □

Any $€ \in \mathcal{J}$ is left infinitely distributive; if $\vee B$ exists for a subset B of €, then $\vee uB$ exists for every $u \in €$ and $\vee uB = u(\vee B)$. The right infinitely distributive is defined dually. An € is infinitely distributive if it is left as well as right infinitely distributive. The concept of 'joins' leads us to finalise the following assertions about £.

Theorem 1. The following statements are equivalent for any $£ \in \mathcal{L}$ and $aa' = a'a$ for all $a \in £$:

i) £ is infinitely distributive;

ii) The set of idempotents of £ is an infinitely distributive semilattice;

iii) For all subsets B and C of £, if $b = \vee B$ and $c = \vee C$, then $\vee BC = (\vee B)(\vee C)$.

Proof. i) ⇒ ii): It follows directly from Lemma 6.

ii) ⇒ iii): Let $B = \{b_i : i \in I\}$ and $C = \{c_j : j \in J\}$ be two subsets of £ so that $b = \vee B$ and $= \vee C$. Obviously, bc is an upper bound of BC. Now we just need to show that $\vee BC = bc$. Let h be another upper bound of BC such that $b_i c_j \le h$ for all $b_i \in B$ and $c_j \in C$. But $b_i \le b$ and $c_j \le c$ such that $(b_i b_i')(c_j c_j') = (b_i c_j)(b_i' c_j') \le h(b_i' c_j')$. Since $E(£)$ is infinitely distributive, therefore $(bb')(c_j c_j') = (((\vee B)(\vee B)')(c_j c_j')) = (\vee (b_i b_i'))(c_j c_j') = \vee ((b_i b_i')(c_j c_j'))$ by Proposition 5. However, $(b_i b_i')(c_j c_j') \le h(b'c')$ implies that $(bb')(c_j c_j') \le h(b'c')$. Since $E(£)$ is infinitely distributive, therefore $(bb')(cc') = (bb')((\vee C)(\vee C)') = (bb')(\vee (c_j c_j')) = \vee ((bb')(c_j c_j'))$ by Proposition 5. Consequently, $(bb')(cc') \le h(b'c')$. This implies that $bc = ((bb')b)((cc')c) =$

$\left((bb')(cc')\right)(bc) \leq \left(h(b'c')\right)(bc) = (h(bc)')(bc) \leq (hh')h$. Hence $bc = \vee BC$.

iii) ⇒ i): It is immediate from (iii). □

CONCLUSIONS

We have related the natural partial order and compatibility relations in a left permutable inverse LA-semigroup. Some fundamental properties of a left permutable inverse LA-semigroup have also been obtained, which can be applied to ascertain Wagner-Preston theorem in a left permutable inverse LA-semigroup. Also, we have furnished the conditions under which a left permutable inverse LA-semigroup is infinitely distributive. Using these conditions, it has also been proved that the set of all permissible subsets of a left permutable inverse LA-semigroup is complete and infinitely distributive.

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