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Full Paper

γ–Independent dominating graphs of paths and cycles

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Abstract: An independent dominating set D of a graph $G = (V(G),E(G))$ is a set of pairwise non-adjacent vertices of G such that every vertex of G not in D is adjacent to at least one vertex in D. The independent domination number of G, denoted by $\gamma_i(G)$, is the minimum cardinality of an independent dominating set of G. An independent dominating set of cardinality $\gamma_i(G)$ is called a $\gamma_i(G)$–set. We introduce the γ–independent dominating graph of G, denoted by $ID_\gamma(G)$, as the graph whose vertex set is the set of all $\gamma_i(G)$–sets, and two $\gamma_i(G)$–sets are adjacent in $ID_\gamma(G)$ if they differ by one vertex. In this paper we present the γ–independent dominating graphs of all paths and all cycles.

Keywords: independent dominating graph, independent dominating set, independent domination number

INTRODUCTION

Let $G = (V(G),E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A set $D \subseteq V(G)$ is a *dominating set* if every vertex not in D is adjacent to some vertex in D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a $\gamma(G)$*–set*. For detailed literature on domination, see Haynes et al. [1, 2].

In 2010 Lakshmanan and Vijayakumar [3] defined a *gamma graph* $\gamma.G$ of G as the graph whose vertex set is the set of all $\gamma(G)$–sets. Two $\gamma(G)$–sets D_1 and D_2 are adjacent in

$\gamma.G$ if $D_1 = D_2 \backslash \{u\} \cup \{v\}$ for some vertices $u \in D_2$ and $v \notin D_2$. They discussed the relationship between the clique number and the independence number of a graph and its gamma graph. Later, Bień [4], Sridharan and Subramanian [5] and Sridharan et al. [6] studied and gave some properties of this gamma graph.

In 2011 Fricke et al. [7] also defined a *gamma graph* $G(\gamma)$ with slightly different meaning. The vertex set of $G(\gamma)$ is the same as one of $\gamma.G$. Two $\gamma(G)$–sets D_1 and D_2 are adjacent in $G(\gamma)$ if $D_1 = D_2 \backslash \{u\} \cup \{v\}$ for some vertices $u \in D_2$ and $v \notin D_2$, and they must be adjacent in G. They considered the structure of $G(\gamma)$ for some graph G. Connelly et al. [8] gave a note on gamma graphs.

Another class of graphs whose vertices correspond to dominating sets was introduced by Haas and Seyffarth in 2014 [9]. They defined a *k–dominating graph* $D_k(G)$ as the graph whose vertex set contains all dominating sets D of G such that $|D| \leq k$. Two dominating sets are adjacent in $D_k(G)$ if one can be obtained from the other by either adding or deleting a single vertex. They provided the conditions that ensure $D_k(G)$ is connected.

In 2017 Wongsriya and Trakultraipruk [10] introduced a *γ–total dominating graph* of a graph G, denoted by $TD_\gamma(G)$, as the graph whose vertices are γ–total dominating sets, and two γ–total dominating sets are adjacent in $TD_\gamma(G)$ if they differ by one vertex. They considered the γ–total dominating graphs of paths and cycles.

An *independent set* of a graph G is a set of pairwise non-adjacent vertices of G. A set $D \subseteq V(G)$ is an *independent dominating set* of G if it is both an independent set and a dominating set of G. The theory of independent domination was formalised by Berge [11] and Ore [12] in 1962. The *independent domination number* of G, denoted by $\gamma_i(G)$, is the minimum cardinality of an independent dominating set of G. An independent dominating set of cardinality $\gamma_i(G)$ is called a *$\gamma_i(G)$–set*. Independent dominating sets and independent domination numbers of graphs are extensively studied in the literature; see for example Allan and Laskar [13] and Topp and Volkmann [14]. We introduce the *γ–independent dominating graph* of G, denoted by $ID_\gamma(G)$, as the graph whose vertex set is the set of all $\gamma_i(G)$–sets, and two $\gamma_i(G)$–sets D_1 and D_2 are adjacent in $ID_\gamma(G)$ if $D_1 = D_2 \backslash \{u\} \cup \{v\}$ for some vertices $u \in D_2$ and $v \notin D_2$. For instance, the γ–independent dominating graphs of the path $P_5 = v_1v_2 \cdots v_5$ and the path $P_7 = v_1v_2 \cdots v_7$ are shown in Figures 1 and 2 respectively.

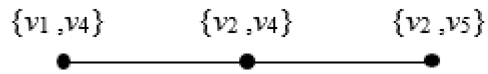

Figure 1. The γ–independent dominating graph of P_5

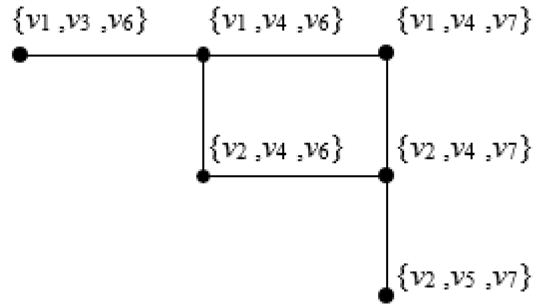

Figure 2. The γ–independent dominating graph of P_7

In this paper we consider the γ–independent dominating graphs of all paths and all cycles. For notations and terminology, we in general follow West [15].

RESULTS

γ–Independent Dominating Graphs of Paths

In this section we consider the γ–independent dominating graphs of paths. Let n be a positive integer. Let $P_n = v_1v_2 \cdots v_n$ be a path with n vertices. The *Cartesian product* of graphs G and H, denoted by $G\square H$, is the graph whose vertex set is $V(G) \times V(H)$, and two vertices (u_1,v_1) and (u_2,v_2) are adjacent in $G\square H$ if $u_1 = u_2$ and v_1 is adjacent to v_2 in H, or $v_1 = v_2$ and u_1 is adjacent to u_2 in G. The *m×n grid graph* is the Cartesian product graph $P_m\square P_n$, whose vertices correspond to the points in the plane with integer coordinates x and y. Let k be a positive integer. For $i, j \in \{1, 2, \dots , k\}$, let $v_{i,j}$ be the vertex at position (i, j) of a $k\times k$ grid graph. We define a *stairgrid* of size k, denoted by S_k, to be the subgraph of $P_k\square P_k$ induced by $\{v_{i,j} \mid 1 \le i \le j \le k\}$. For instance, the stairgrids S_1, S_2, S_3 and S_k are shown in Figure 3.

Goddard and Henning [16] provided the independent domination numbers of paths and cycles, which are shown in the following proposition.

Proposition 1. Let $n \ge 3$ be an integer. Then $\gamma_i(P_n) = \gamma_i(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Let n be a positive integer. We consider the γ–independent dominating graph of a path with n vertices in three cases. If n is divisible by three, then the γ–independent dominating graph of P_n contains only one vertex. If $n = 3k+2$ for some non-negative integer k, then the γ–independent dominating graph of P_n is a path with $k+2$ vertices. Finally if $n = 3k+1$ for some non-negative integer k, then the γ–independent dominating graph of P_n is a stairgrid of size $k+1$.

Theorem 1. Let $k \ge 1$ be an integer. Then $ID_\gamma(P_{3k}) \cong K_1$.

Proof. Let $P_{3k} = v_1v_2 \cdots v_{3k}$ be a path with $3k$ vertices. Each dominating vertex in a path can dominate at most three vertices. By Proposition 1, $\gamma_i(P_{3k}) = k$. Then each vertex in a $\gamma_i(P_{3k})$–set must dominate exactly three different vertices. Thus, there is only one $\gamma_i(P_{3k})$–set which is $\{v_2, v_5, \dots , v_{3k-1}\}$. This completes the proof.

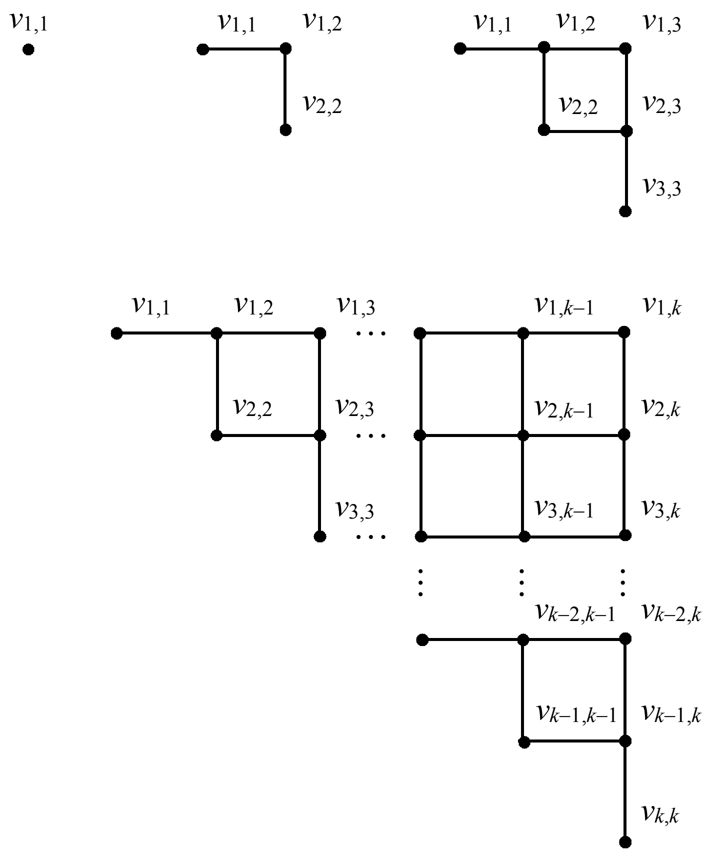

Figure 3. The stairgrids S_1, S_2, S_3, and S_k respectively

Next, we give some properties of a $\gamma_i(P_{3k+2})$–set to study the γ–independent dominating graph of a path with $3k$+2 vertices.

Lemma 1. Let $k \geq 0$ be an integer. Then there is only one $\gamma_i(P_{3k+2})$–set that contains v_{3k+2} and the only one $\gamma_i(P_{3k+2})$–set that contains v_1. Moreover, both sets are of degree one in $ID_\gamma(P_{3k+2})$.

Proof. By Proposition 1, we get $\gamma_i(P_{3k+2}) = k+1$. Since v_{3k+2} dominates v_{3k+2} and v_{3k+1}, the other k dominating vertices in this $\gamma_i(P_{3k+2})$–set must dominate $v_1, v_2, \ldots, v_{3k}$. We may consider these $3k$ vertices as a path with $3k$ vertices. Since $\gamma_i(P_{3k}) = k$, these k dominating vertices form a $\gamma_i(P_{3k})$–set. By Theorem 1, D = $\{v_2, v_5, \ldots, v_{3k-1}\}$ is the unique $\gamma_i(P_{3k})$–set. Hence $X = D \cup \{v_{3k+2}\}$ is the only one $\gamma_i(P_{3k+2})$–set containing v_{3k+2}. Next, we show that the degree of X in $ID_\gamma(P_{3k+2})$ is one. Since each $\gamma_i(P_{3k+2})$–set must contain either v_{3k+2} or v_{3k+1}, the other $\gamma_i(P_{3k+2})$–sets of X must contain v_{3k+1}. Hence $D \cup \{v_{3k+1}\}$ is the only one neighbour of X in $ID_\gamma(P_{3k+2})$, so the degree of X is one. Similarly, there is only one $\gamma_i(P_{3k+2})$–set that contains v_1, and its degree in $ID_\gamma(P_{3k+2})$ is one.

Theorem 2. Let $k \geq 0$ be an integer. Then $ID_\gamma(P_{3k+2}) \cong P_{k+2}$.

Proof. Let $P_{3k+2} = v_1v_2 \cdots v_{3k+2}$ be a path with $3k$+2 vertices. We prove by induction on k. It is easy to see that there are two $\gamma_i(P_2)$–sets, which are $\{v_1\}$ and $\{v_2\}$, so $ID_\gamma(P_2) \cong P_2$. We assume that $ID_\gamma(P_{3k+2}) \cong P_{k+2} \cong D_1D_2 \cdots D_{k+2}$, where D_i is a $\gamma_i(P_{3k+2})$–set for all i. By Lemma 1, without loss of generality, we may assume that D_{k+2} contains v_{3k+2}, and the other $\gamma_i(P_{3k+2})$–sets contain v_{3k+1}. We prove that $ID_\gamma(P_{3k+5}) \cong P_{k+3}$. Recall that $P_{3k+5} = v_1v_2 \cdots v_{3k+2}v_{3k+3}v_{3k+4}v_{3k+5}$ and $\gamma_i(P_{3k+5}) = k+2$. Clearly, each $\gamma_i(P_{3k+5})$–set cannot contain all of v_{3k+3}, v_{3k+4} and v_{3k+5}. Next, we show that the $\gamma_i(P_{3k+5})$–set must contain only one vertex from them. Suppose for a contradiction there is a $\gamma_i(P_{3k+5})$–set that contains two vertices from them, so they are v_{3k+3} and v_{3k+5}. Thus, these two vertices dominate v_{3k+2}, v_{3k+3}, v_{3k+4} and v_{3k+5}. Hence the other k dominating vertices in this $\gamma_i(P_{3k+5})$–set must dominate at least $3k+1$ vertices. This contradicts the fact that k vertices can dominate at most $3k$ vertices on the path. Since each $\gamma_i(P_{3k+5})$–set must dominate v_{3k+5}, it contains one vertex from $\{v_{3k+4}, v_{3k+5}\}$. The other $k+1$ dominating vertices must dominate $v_1, v_2, \ldots, v_{3k+2}$. Since $\gamma_i(P_{3k+2}) = k+1$, these $k+1$ dominating vertices form a $\gamma_i(P_{3k+2})$–set. Hence each $\gamma_i(P_{3k+5})$–set is a union of a $\gamma_i(P_{3k+2})$–set and a vertex from $\{v_{3k+4}, v_{3k+5}\}$. By the induction hypothesis, there are $k+2$ $\gamma_i(P_{3k+2})$–sets, which are $D_1, D_2, \ldots, D_{k+2}$. We first consider all $\gamma_i(P_{3k+5})$–sets that contain v_{3k+4}. For each $i \in \{1, 2, \ldots, k+2\}$, let $X_i = D_i \cup \{v_{3k+4}\}$. Thus, these X_i's are all $\gamma_i(P_{3k+5})$–sets that contain v_{3k+4}, and they form a path $X_1X_2 \cdots X_{k+2}$ in $ID_\gamma(P_{3k+5})$. Note that D_{k+2} is only $\gamma_i(P_{3k+2})$–set containing v_{3k+2}. Then the set $X_{k+3} = D_{k+2} \cup \{v_{3k+5}\}$ is the unique $\gamma_i(P_{3k+5})$–set that contains v_{3k+5}, and it is adjacent to only X_{k+2} in $ID_\gamma(P_{3k+5})$. This completes the proof.

Observation 1. Each $\gamma_i(P_{3k+5})$–set can be written as a union of a $\gamma_i(P_{3k+2})$–set and a vertex from $\{v_{3k+4}, v_{3k+5}\}$.

In Theorem 3, we present the γ–independent dominating graph of a path with $3k+1$ vertices, where k is a non-negative integer. To prove this, we use the following lemma and observation.

Lemma 2. Let $k \geq 1$ be an integer. Then there are $k+1$ $\gamma_i(P_{3k+1})$–sets that contain v_{3k+1}, and they form a path in $ID_\gamma(P_{3k+1})$. Moreover, on this path the internal vertices are of degree three, one end-vertex is of degree one, and the other end-vertex is of degree two. The same results hold for the $\gamma_i(P_{3k+1})$–sets that contain v_1.

Proof. Note that $\gamma_i(P_{3k+1}) = k+1$. If v_{3k+1} is in a $\gamma_i(P_{3k+1})$–set, the other k dominating vertices must dominate $v_1, v_2, \ldots, v_{3k-1}$. We may consider these $3k-1$ vertices as a path with $3k-1$ vertices. Since $\gamma_i(P_{3k-1}) = k$, these k dominating vertices form a $\gamma_i(P_{3k-1})$–set. Thus, such a $\gamma_i(P_{3k+1})$–set is a union of a $\gamma_i(P_{3k-1})$–set and $\{v_{3k+1}\}$. By Theorem 2, $ID_\gamma(P_{3k-1}) \cong P_{k+1} \cong X_1X_2 \cdots X_{k+1}$, where X_i is a $\gamma_i(P_{3k-1})$–set for all i. For each $i \in \{1, 2, \ldots, k+1\}$, let $Y_i = X_i \cup \{v_{3k+1}\}$. Then $Y_1, Y_2, \ldots, Y_{k+1}$ are all $\gamma_i(P_{3k+1})$–sets that contain v_{3k+1}, and they form a path $Y_1Y_2 \cdots Y_{k+1}$ in $ID_\gamma(P_{3k+1})$. Note that in $ID_\gamma(P_{3k+1})$, the vertices Y_1 and Y_{k+1} have the only neighbour containing v_{3k+1}, and the internal vertices $Y_2, Y_3, \ldots, Y_k$ have two neighbours containing v_{3k+1}. Since each $\gamma_i(P_{3k+1})$–set contains either v_{3k+1} or v_{3k}, the other neighbours of $Y_i = X_i \cup \{v_{3k+1}\}$ must contain v_{3k}. By Lemma 1, there is only one $\gamma_i(P_{3k-1})$–set that contains v_{3k-1}. Assume that X_{k+1} contains v_{3k-1}, so $X_1, X_2, \ldots, X_k$ contain v_{3k-2}. Hence $Y_{k+1} = X_{k+1} \cup \{v_{3k+1}\}$ has no neighbour containing v_{3k}, and it has degree one in $ID_\gamma(P_{3k+1})$. Furthermore, for each $i \in \{1, 2, \ldots, k\}$, the set $X_i \cup \{v_{3k}\}$ is the only neighbour of Y_i containing v_{3k}. Then Y_1 has degree two and $Y_2, Y_3, \ldots, Y_k$ are of degree three in $ID_\gamma(P_{3k+1})$.

Observation 2. Each $\gamma_i(P_{3k+1})$–set that contains v_{3k+1} can be written as a union of a $\gamma_i(P_{3k-1})$–set and $\{v_{3k+1}\}$.

Theorem 3. Let $k \geq 0$ be an integer. Then $ID_\gamma(P_{3k+1}) \cong S_{k+1}$.

Proof. Let $P_{3k+1} = v_1v_2 \cdots v_{3k+1}$ be a path with $3k+1$ vertices. We prove by induction on k. There is only one $\gamma_i(P_1)$–set which is $\{v_1\}$, so $ID_\gamma(P_1) \cong P_1 \cong S_1$. Since there are three $\gamma_i(P_4)$–sets which are $\{v_1,v_3\}$, $\{v_1,v_4\}$ and $\{v_2,v_4\}$, $ID_\gamma(P_4) \cong P_3 \cong S_2$. Let $k \geq 1$. We assume that $ID_\gamma(P_{3k+1}) \cong S_{k+1}$. For $1 \leq i \leq j \leq k+1$, let $Y_{i,j}$ be the $\gamma_i(P_{3k+1})$–set at position (i, j) of S_{k+1} as shown in Figure 4.

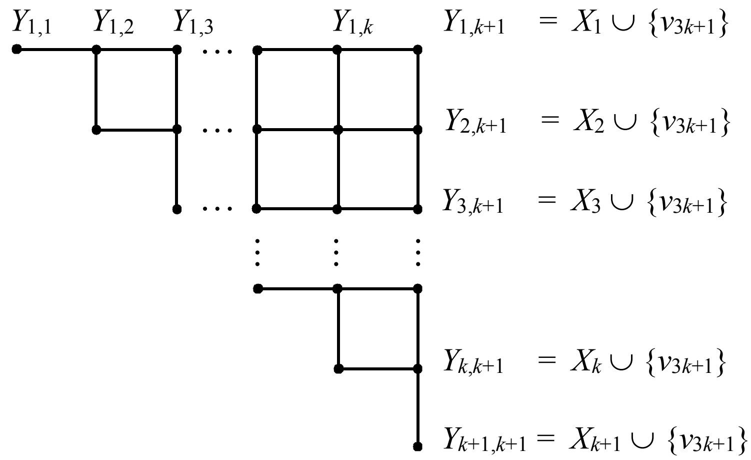

Figure 4. The γ–independent dominating graph of P_{3k+1}

We see that the vertices in the first row and ones in the last column of S_{k+1} form the paths with $k+1$ vertices. By Lemma 2, without loss of generality, we may assume that the vertices in the last column of S_{k+1} are the $\gamma_i(P_{3k+1})$–sets that contain v_{3k+1}. By Observation 2, for each $i \in \{1, 2, \ldots, k+1\}$, the set $Y_{i,k+1} = X_i \cup \{v_{3k+1}\}$, where X_i is a $\gamma_i(P_{3k-1})$–set. Furthermore, the vertices in the first row of S_{k+1} are the $\gamma_i(P_{3k+1})$–sets that contain v_1. Since $Y_{1,k+1} = X_1 \cup \{v_{3k+1}\}$, the set X_1 contains v_1. By Lemma 1, the set X_{k+1} contains v_{3k-1}.

We prove that $ID_\gamma(P_{3k+4}) \cong S_{k+2}$, where $P_{3k+4} = v_1v_2 \cdots v_{3k+1}v_{3k+2}v_{3k+3}v_{3k+4}$. Note that $\gamma_i(P_{3k+4}) = k+2$, and each $\gamma_i(P_{3k+4})$–set contains exactly one of v_{3k+3} and v_{3k+4}. We first consider all $\gamma_i(P_{3k+4})$–sets that contain v_{3k+3}. Since v_{3k+3} dominates v_{3k+2}, v_{3k+3} and v_{3k+4}, the other $k+1$ dominating vertices in this $\gamma_i(P_{3k+4})$–set must dominate $v_1, v_2, \ldots, v_{3k+1}$. Since $\gamma_i(P_{3k+1}) = k+1$, these $k+1$ dominating vertices form a $\gamma_i(P_{3k+1})$–set. Then such a $\gamma_i(P_{3k+4})$–set is a union of a $\gamma_i(P_{3k+1})$–set and $\{v_{3k+3}\}$. By the induction hypothesis, all $\gamma_i(P_{3k+1})$–sets form a stairgrid of size $k+1$. For $1 \leq i \leq j \leq k+1$, let $Y'_{i,j} = Y_{i,j} \cup \{v_{3k+3}\}$. Then these $Y'_{i,j}$'s are all $\gamma_i(P_{3k+4})$–sets that contain v_{3k+3}, and they form the stairgrid having the same size.

Next, we consider all $\gamma_i(P_{3k+4})$–sets that contain v_{3k+4}. By Lemma 2, there are $k+2$ $\gamma_i(P_{3k+4})$–sets that contain v_{3k+4}, and they form a path in $ID_\gamma(P_{3k+4})$. Recall that in the last column of $ID_\gamma(P_{3k+1})$, for each $i \in \{1, 2, \ldots, k+1\}$, the set $Y_{i,k+1}$ is a $\gamma_i(P_{3k+1})$–set that contains v_{3k+1}. We let $Y'_{i,k+2} = Y_{i,k+1} \cup \{v_{3k+4}\}$ for all i. Then we have $k+1$ $\gamma_i(P_{3k+4})$–sets containing v_{3k+4}, and these sets form a path $Y'_{1,k+2}\, Y'_{2,k+2} \cdots Y'_{k+1,k+2}$. Furthermore, for each $i \in \{1, 2, \ldots , k + 1\}$, the set $Y'_{i,k+2}$ is adjacent to $Y'_{i,k+1}$ in $ID_\gamma(P_{3k+4})$. Next, we construct the remaining $\gamma_i(P_{3k+4})$–set that contains v_{3k+4}. Recall that in $ID_\gamma(P_{3k+1})$, $Y_{k+1,k+1} = X_{k+1} \cup \{v_{3k+1}\}$, where X_{k+1} is the unique $\gamma_i(P_{3k-1})$–set that contains v_{3k-1}. Then the set $Y'_{k+1,k+2} = Y_{k+1,k+1} \cup \{v_{3k+4}\} = X_{k+1} \cup \{v_{3k+1}\} \cup \{v_{3k+4}\}$ contains v_{3k-1}, v_{3k+1} and v_{3k+4}. Let $Y'_{k+2,k+2} = X_{k+1} \cup \{v_{3k+2}\} \cup \{v_{3k+4}\}$, so it is another $\gamma_i(P_{3k+4})$–set that contains v_{3k+4} and it is adjacent to $Y'_{k+1,k+2}$. Since each $\gamma_i(P_{3k+4})$–set contains either v_{3k+3} or v_{3k+4}, the set $Y'_{k+2,k+2} = X_{k+1} \cup \{v_{3k+2}\} \cup \{v_{3k+4}\}$ has no neighbours that contain v_{3k+3}. This completes the proof.

γ–Independent Dominating Graphs of Cycles

In this section we consider the γ–independent dominating graphs of cycles. Let k be a positive integer. For $i\,,\, j \in \{1, 2, \ldots , 2k+1\}$, let $v_{i,j}$ be the vertex at position (i, j) of $P_{2k+1}\square P_{2k+1}$. We define a *twisting stair* of size k, denoted by T_k, to be the graph with the following three properties.

(i) The vertex set $V(T_k)$ is the set of all vertices $v_{i,j}$ in $P_{2k+1}\square P_{2k+1}$ such that $0 \le j - i \le k-1$.

(ii) The edge set $E(T_k)$ contains all edges in $P_{2k+1}\square P_{2k+1}$ that have both end points in $V(T_k)$.

(iii) For all $i \in \{1, 2, \ldots , k\}$, the vertices $v_{1,i}$ and $v_{i+k+1,2k+1}$ are the same.

For instance, the twisting stairs T_1, T_2 and T_4 are shown in Figure 5.

For an integer $n \ge 3$, we let $C_n = v_0v_1 \cdots v_{n-1}v_0$ be a cycle with n vertices. It is easy to see that there are three $\gamma_i(C_3)$–sets, which are $\{v_0\}$, $\{v_1\}$ and $\{v_2\}$, so $ID_\gamma(C_3) \cong C_3$. We next consider the γ–independent dominating graphs of cycles with $n \ge 4$ vertices in three cases. If n is divisible by three, then the γ–independent dominating graph of C_n contains only three isolated vertices. If $n = 3k+1$ for some positive integer k, then the γ–independent dominating graph of C_n is a twisting stair of size k. Finally if $n = 3k+2$ for some positive integer k, then the γ–independent dominating graph of C_n is a cycle with $3k+2$ vertices.

Theorem 4. Let $k \ge 2$ be an integer. Then $ID_\gamma(C_{3k}) \cong 3K_1$.

Proof. Let $C_{3k} = v_0v_1 \cdots v_{3k-1}v_0$ be a cycle with $3k$ vertices. Note that $\gamma_i(C_{3k}) = k$. Then each dominating vertex must dominate exactly three different vertices of the cycle. For each $i \in \{0, 1, 2\}$, let $D_i = \{v_{3m+i} \mid 0 \le m \le k-1\}$. It is clear that they are the only $\gamma_i(C_{3k})$–sets. Since they are pairwise disjoint, $ID_\gamma(C_{3k}) \cong 3K_1$.

Before we prove Theorem 5, we provide some notation and some properties that we use in the proof. For a positive integer n and a non-negative integer i, we define $P_n(v_i : v_{i+n-1})$ to be the path $v_iv_{i+1} \cdots v_{i+n-1}$, and the path $P_n(v_1 : v_n)$ is always denoted by P_n.

Let $k \ge 3$ be a positive integer. By Theorem 3, we have $ID_\gamma(P_{3k-2}) \cong S_k$. Let $Y_{i,j}$ be the $\gamma_i(P_{3k-2}(v_a : v_b))$–set at position (i, j) in S_k, where a and $b = a + 3k-3$ are non-negative integers. By Lemma 2, without loss of generality, we may assume that all $\gamma_i(P_{3k-2}(v_a : v_b))$–sets in the first row of this S_k contain v_a and all $\gamma_i(P_{3k-2}(v_a : v_b))$–sets in the last column contain v_b. Furthermore, we can apply Observation 2 to derive $Y_{1,j} = \{v_a\} \cup X_j$, where X_j is a $\gamma_i(P_{3k-4}(v_{a+2} : v_b))$–set and $Y_{i,k} = \bar{X}_i \cup \{v_b\}$, where $\bar{X}_i$ is a $\gamma_i(P_{3k-4}(v_a : v_{b-2}))$–set for each $i, j \in \{1, 2, \ldots , k\}$. Since the set $Y_{1,k}$ is both in the first row and the last column of S_k, we have $Y_{1,k} = \{v_a\} \cup X_k = \bar{X}_1 \cup \{v_b\}$. Hence X_k must contain v_b, and $\bar{X}_1$ must contain v_a. By Lemma 1, only X_k contains v_b and only X_1 contains v_{a+2}. Similarly, only $\bar{X}_1$ contains v_a and only $\bar{X}_k$ contains v_{b-2}. To summarise, we have the following properties:

(p1) for each $j \in \{1, 2, \dots, k\}$, the set $Y_{1,j} = \{v_a\} \cup X_j$, where X_j is a $\gamma_i(P_{3k-4}(v_{a+2} : v_b))$–set such that only X_1 contains v_{a+2} and only X_k contains v_b;

(p2) for all $i \neq 1$, the set $Y_{i,j}$ contains v_{a+1};

(p3) for each $i \in \{1, 2, \dots, k\}$, the set $Y_{i,k} = \bar{X}_i \cup \{v_b\}$, where $\bar{X}_i$ is a $\gamma_i(P_{3k-4}(v_a : v_{b-2}))$–set such that only $\bar{X}_1$ contains v_a and only $\bar{X}_k$ contains v_{b-2}; and

(p4) for all $j \neq k$, the set $Y_{i,j}$ contains v_{b-1}.

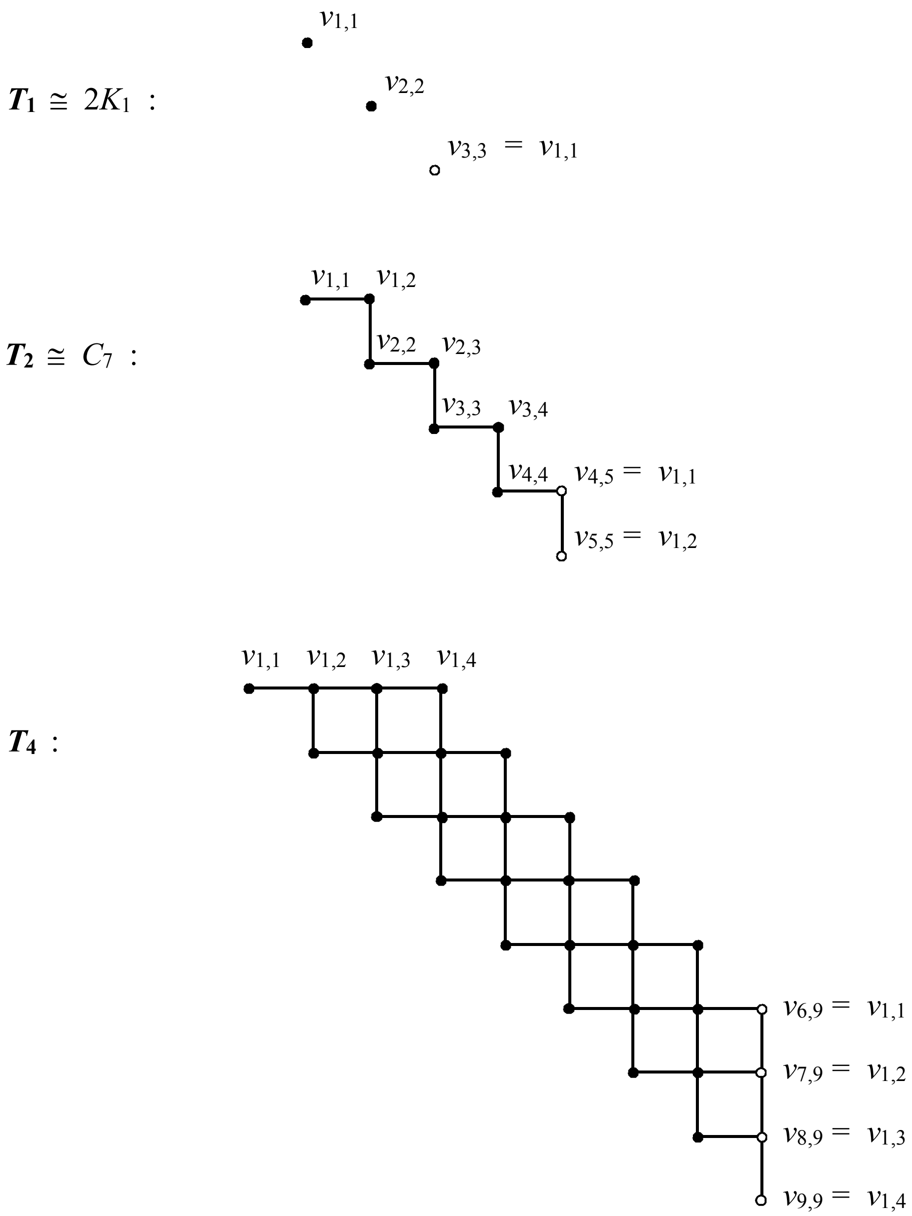

Figure 5. The twisting stairs T_1, T_2 and T_4

Theorem 5. Let $k \geq 1$ be an integer. Then $ID_\gamma(C_{3k+1}) \cong T_k$.

Proof. Let $C_{3k+1} = v_0v_1 \cdots v_{3k}v_0$ be a cycle with $3k+1$ vertices. For $k = 1$, there are two $\gamma_i(C_4)$–sets, which are $\{v_1,v_3\}$ and $\{v_2,v_4\}$, so $ID_\gamma(C_4) \cong 2K_1 \cong T_1$. For $k = 2$, there are seven $\gamma_i(C_7)$–sets, which are $\{v_1,v_3,v_5\}$, $\{v_1,v_3,v_6\}$, $\{v_1,v_4,v_6\}$, $\{v_2,v_4,v_6\}$, $\{v_2,v_4,v_7\}$, $\{v_2,v_5,v_7\}$ and $\{v_3,v_5,v_7\}$, so $ID_\gamma(C_7) \cong C_7 \cong T_2$. Let $k \geq 3$. Since each $\gamma_i(C_{3k+1})$–set has to dominate the vertex v_0, it contains either v_{3k}, v_0 or v_1. First, we consider all $\gamma_i(C_{3k+1})$–sets that contain v_{3k}. Note that $\gamma_i(C_{3k+1}) = k+1$. If v_{3k} is in a $\gamma_i(C_{3k+1})$–set, the other k dominating vertices in this $\gamma_i(C_{3k+1})$–set must dominate $v_1, v_2, \ldots, v_{3k-2}$. Since $\gamma_i(P_{3k-2}) = k$, these k dominating vertices form a $\gamma_i(P_{3k-2})$–set. Thus, such a $\gamma_i(C_{3k+1})$–set is a union of a $\gamma_i(P_{3k-2})$–set and $\{v_{3k}\}$. By Theorem 3, $ID_\gamma(P_{3k-2}) \cong S_k$. For all i, j, let $Y_{i,j}$ be the $\gamma_i(P_{3k-2})$–set at position (i, j) in S_k, and $Z_{i,j}^{(1)} = Y_{i,j} \cup \{v_{3k}\}$. Then $Z_{i,j}^{(1)}$'s are all $\gamma_i(C_{3k+1})$–sets that contain v_{3k}, and they form a stairgrid of size k in $ID_\gamma(C_{3k+1})$. Let $S_k^{(1)}$ be the subgraph of $ID_\gamma(C_{3k+1})$ induced by these $Z_{i,j}^{(1)}$'s. Similarly, all $\gamma_i(C_{3k+1})$–sets that contain v_0 form a stairgrid of size k. Let $Z_{i,j}^{(2)}$ be the $\gamma_i(C_{3k+1})$–set that contains v_0 at position (i, j), and $S_k^{(2)}$ the subgraph of $ID_\gamma(C_{3k+1})$ induced by these $Z_{i,j}^{(2)}$'s. We also let $Z_{i,j}^{(3)}$ be the $\gamma_i(C_{3k+1})$–set that contains v_1 at position (i, j), and $S_k^{(3)}$ the subgraph of $ID_\gamma(C_{3k+1})$ induced by these $Z_{i,j}^{(3)}$'s. The subgraphs $S_k^{(1)}$, $S_k^{(2)}$ and $S_k^{(3)}$ of $ID_\gamma(C_{3k+1})$ are shown in Figure 6.

Since each $\gamma_i(C_{3k+1})$–set cannot contain both v_{3k} and v_0, the subgraphs $S_k^{(1)}$ and $S_k^{(2)}$ do not have any vertices in common. Similarly, $S_k^{(2)}$ and $S_k^{(3)}$ do not share any vertices. Next, we consider all $\gamma_i(C_{3k+1})$–sets that are in both $S_k^{(1)}$ and $S_k^{(3)}$. These sets must contain v_{3k} and v_1. Recall that

$Z_{i,j}^{(1)} = \{v_{3k}\} \cup Y_{i,j}^{(1)}$, where $Y_{i,j}^{(1)}$ is a $\gamma_i(P_{3k-2}(v_1 : v_{3k-2}))$–set, and

$Z_{s,t}^{(3)} = \{v_1\} \cup Y_{s,t}^{(3)}$, where $Y_{s,t}^{(3)}$ is a $\gamma_i(P_{3k-2}(v_3 : v_{3k}))$–set.

Then we consider the set $Y_{i,j}^{(1)}$ that contains v_1 and $Y_{s,t}^{(3)}$ that contains v_{3k}. By ($p1$), we have $Z_{1,j}^{(1)} = \{v_{3k}\} \cup Y_{1,j}^{(1)} = \{v_{3k}\} \cup \{v_1\} \cup X_j^{(1)}$, where $X_j^{(1)}$ is a $\gamma_i(P_{3k-4}(v_3 : v_{3k-2}))$–set for all $j \in \{1, 2, \ldots, k\}$, and only $X_1^{(1)}$ contains v_3. By ($p3$), we have $Z_{s,t}^{(3)} = \{v_1\} \cup Y_{s,t}^{(3)} = \{v_1\} \cup \bar{X}_s^{(3)} \cup \{v_{3k}\}$, where $\bar{X}_s^{(3)}$ is a $\gamma_i(P_{3k-4}(v_3 : v_{3k-2}))$–set for all $s \in \{1, 2, \ldots, k\}$, and only $\bar{X}_1^{(3)}$ contains v_3. Thus, $X_1^{(1)} = \bar{X}_1^{(3)}$. By Theorem 2, $X_1^{(1)}X_2^{(1)} \cdots X_k^{(1)} \cong ID_\gamma(P_{3k-4}(v_3 : v_{3k-2})) \cong \bar{X}_1^{(3)}\bar{X}_2^{(3)} \cdots \bar{X}_k^{(3)}$. Hence $X_j^{(1)} = \bar{X}_j^{(3)}$ for all j. Thus, $Z_{1,j}^{(1)} = \{v_{3k}, v_1\} \cup X_j^{(1)} = \{v_1\} \cup \bar{X}_j^{(3)} \cup \{v_{3k}\} = Z_{j,k}^{(3)}$ for all $j \in \{1, 2, \ldots, k\}$. Furthermore, the other $\gamma_i(C_{3k+1})$–sets in $S_k^{(1)}$ and ones in $S_k^{(3)}$ cannot contain both v_1 and v_{3k}, so they are mutually different.

Next, we consider all edges between the $\gamma_i(C_{3k+1})$–sets in $S_k^{(1)}$ and ones in $S_k^{(3)}$. Recall that $Z_{1,j}^{(1)} = Z_{j,k}^{(3)}$ for all $j \in \{1, 2, \ldots, k\}$. By ($p2$), for all $i \neq 1$, the set $Y_{i,j}^{(1)}$ contains v_2, so $Z_{i,j}^{(1)}$ contains v_{3k} and v_2. By ($p4$), for all $t \neq k$, the set $Y_{s,t}^{(3)}$ contains v_{3k-1}, so $Z_{s,t}^{(3)}$ contains v_1 and v_{3k-1}. Hence those sets in $S_k^{(1)}$ and those sets in $S_k^{(3)}$ are non-adjacent.

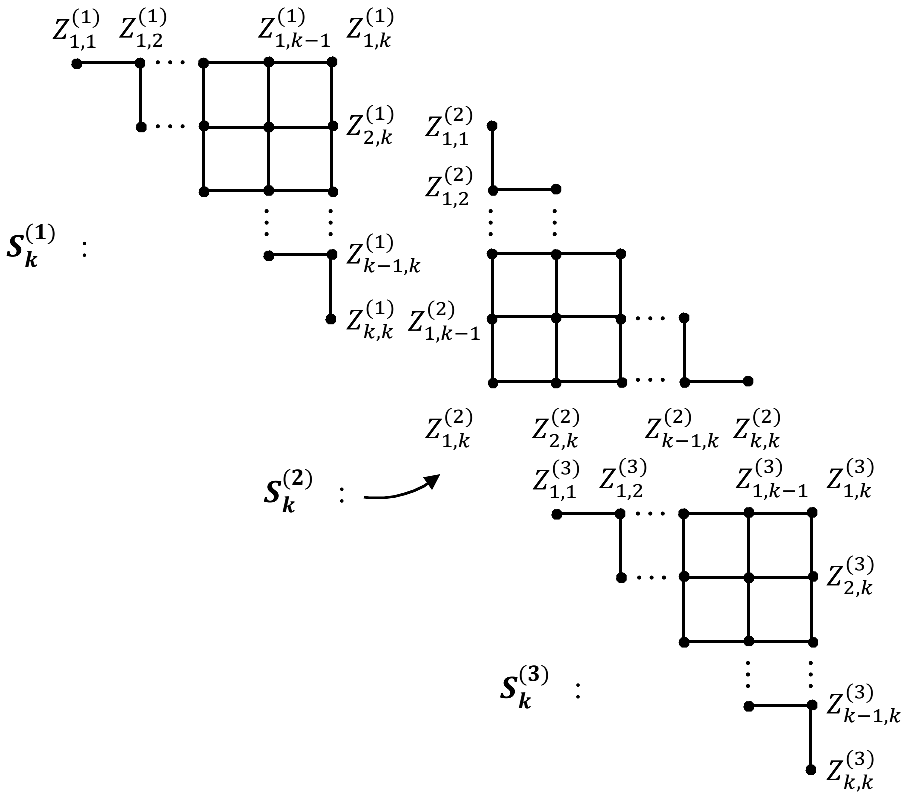

Figure 6. The subgraphs $S_k^{(1)}$, $S_k^{(2)}$ and $S_k^{(3)}$ of $ID_\gamma(C_{3k+1})$

We then consider all edges between the $\gamma_i(C_{3k+1})$–sets in $S_k^{(1)}$and ones in $S_k^{(2)}$. Note that each $\gamma_i(C_{3k+1})$–set in $S_k^{(1)}$contains v_{3k}, and one in $S_k^{(2)}$contains v_0. Then they are adjacent if and only if they have k dominating vertices in common. Recall that

$Z_{i,j}^{(1)} = \{v_{3k}\} \cup Y_{i,j}^{(1)}$, where $Y_{i,j}^{(1)}$ is a $\gamma_i(P_{3k-2}(v_1 : v_{3k-2}))$–set, and
$Z_{s,t}^{(2)} = \{v_0\} \cup Y_{s,t}^{(2)}$, where $Y_{s,t}^{(2)}$ is a $\gamma_i(P_{3k-2}(v_2 : v_{3k-1}))$–set.

Then $Z_{i,j}^{(1)}$ and $Z_{s,t}^{(2)}$ are adjacent if and only if $Y_{i,j}^{(1)} = Y_{s,t}^{(2)}$. Since $Y_{i,j}^{(1)}$ is a $\gamma_i(P_{3k-2}(v_1 : v_{3k-2}))$–set, it contains one of v_1 and v_2, and one of v_{3k-3} and v_{3k-2}. Similarly, $Y_{s,t}^{(2)}$ contains one of v_2 and v_3, and one of v_{3k-2} and v_{3k-1}. Thus, we only consider $Y_{i,j}^{(1)}$ and $Y_{s,t}^{(2)}$ that contain v_2 and v_{3k-2}. By the properties (p2) and (p3), $Y_{i,j}^{(1)}$ contains v_2 if $i \neq 1$, and it contains v_{3k-2} if $j = k$. Then for $i \in \{2, 3, \ldots , k\}$, the set $Y_{i,k}^{(1)} = \bar{X}_i^{(1)} \cup \{v_{3k-2}\}$, where $\bar{X}_i^{(1)}$ is a $\gamma_i(P_{3k-4}(v_1 : v_{3k-4}))$–set that contains v_2. By (p1) and (p4), $Y_{s,t}^{(2)}$ contains v_2 if $s = 1$, and it contains v_{3k-2} if $t \neq k$. Then for $t \in \{1, 2, \ldots , k-1\}$, the set $Y_{1,t}^{(2)} = \{v_2\} \cup X_t^{(2)}$, where $X_t^{(2)}$ is a $\gamma_i(P_{3k-4}(v_4 : v_{3k-1}))$–set that contains v_{3k-2}. We can apply Observation 1 to $\bar{X}_i^{(1)}$ and $X_t^{(2)}$, so we have

for $i \in \{2, 3, \ldots , k\}$,
$Y_{i,k}^{(1)} = \bar{X}_i^{(1)} \cup \{v_{3k-2}\}$, where $\bar{X}_i^{(1)}$ is a $\gamma_i(P_{3k-4}(v_1 : v_{3k-4}))$–set
$= \left(\{v_2\} \cup \bar{D}_i^{(1)}\right) \cup \{v_{3k-2}\}$, where $\bar{D}_i^{(1)}$ is a $\gamma_i(P_{3k-7}(v_4 : v_{3k-4}))$–set, and
for $t \in \{1, 2, \ldots , k-1\}$,
$Y_{1,t}^{(2)} = \{v_2\} \cup X_t^{(2)}$, where $X_t^{(2)}$ is a $\gamma_i(P_{3k-4}(v_4 : v_{3k-1}))$–set
$= \{v_2\} \cup \left(D_t^{(2)} \cup \{v_{3k-2}\}\right)$, where $D_t^{(2)}$ is a $\gamma_i(P_{3k-7}(v_4 : v_{3k-4}))$–set.

Then the set $Y_{i,k}^{(1)} = Y_{1,t}^{(2)}$ when $\overline{D}_i^{(1)} = D_t^{(2)}$. By Theorem 2, we get $\overline{D}_2^{(1)}\overline{D}_3^{(1)}\cdots\ \overline{D}_k^{(1)} \cong ID_\gamma(P_{3k-7}(v_4 : v_{3k-4})) \cong D_1^{(2)}D_2^{(2)}\cdots D_{k-1}^{(2)}$. By ($p$3), the set $\overline{X}_k^{(1)} = \{v_2\}\cup \overline{D}_k^{(1)}$ contains v_{3k-4}, so $\overline{D}_k^{(1)}$contains v_{3k-4}. Thus, $\overline{D}_2^{(1)}$contains v_4. Similarly, $D_1^{(2)}$ contains v_4 by (p1). By Lemma 1, there is only one $\gamma_i(P_{3k-7}(v_4 : v_{3k-4}))$–set containing v_4, so $\overline{D}_2^{(1)} = D_1^{(2)}$. Hence $\overline{D}_i^{(1)} = D_{i-1}^{(2)}$ for all $i \in \{2, 3, \ldots, k\}$. Then $Y_{i,k}^{(1)} = \{v_2\} \cup \overline{D}_i^{(1)} \cup \{v_{3k-2}\} = \{v_2\} \cup D_{i-1}^{(2)} \cup \{v_{3k-2}\} = Y_{1,i-1}^{(2)}$, so $Z_{i,k}^{(1)}$ and $Z_{1,i-1}^{(2)}$ are adjacent in $ID_\gamma(C_{3k+1})$ for all $i \in \{2, 3, \ldots, k\}$.

Similarly, $Z_{i,k}^{(2)}$ and $Z_{1,i-1}^{(3)}$ are adjacent in $ID_\gamma(C_{3k+1})$ for all $i \in \{2, 3, \ldots, k\}$. This completes the proof.

Theorem 6. Let $k \geq 1$ be an integer. Then $ID_\gamma(C_{3k+2}) \cong C_{3k+2}$.

Proof. Let $C_{3k+2} = v_0v_1 \cdots v_{3k+1}v_0$ be a cycle with $3k$+2 vertices. For each $i \in \{0, 1, \ldots, 3k+1\}$, let $D_i = \{v_{3m+i+1 \pmod{3k+2}} \mid 0 \leq m \leq k\}$. It is easy to check that D_i is a $\gamma_i(C_{3k+2})$–set such that v_i is the only vertex dominated by two dominating vertices in D_i. For instance, the $\gamma_i(C_{3k+2})$–sets D_0 and D_3 are shown in Figure 7.

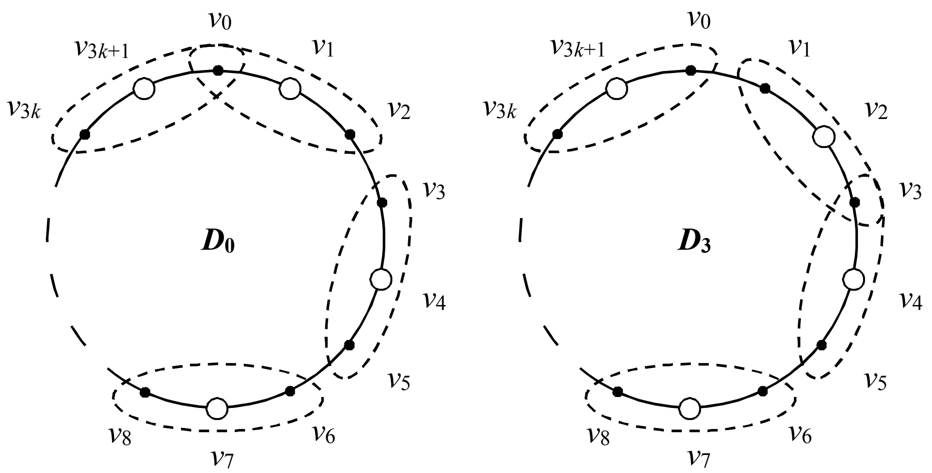

Figure 7. The $\gamma_i(C_{3k+2})$–sets D_0 and D_3 containing white vertices

We next prove that $D_0, D_1, \ldots, D_{3k+1}$ are the only $\gamma_i(C_{3k+2})$–sets. Let D be any $\gamma_i(C_{3k+2})$–set. Then $|D| = \gamma_i(C_{3k+2}) = k+1$. Note that each dominating vertex dominates three vertices. Then these k+1 dominating vertices in D can dominate at most $3k$+3. Since the cycle contains only $3k$+2 vertices, there is a unique vertex v_i in C_{3k+2} dominated by two dominating vertices in D. Hence $D = D_i$.

Let $i \in \{0, 1, \ldots, 3k+1\}$. We consider all neighbours of D_i in $ID_\gamma(C_{3k+2})$. Note that v_{i-1} and v_{i+1} in D_i dominate v_i. Then we can only replace $v_{i-1 \pmod{3k+2}}$ by $v_{i-2 \pmod{3k+2}}$ or replace $v_{i+1 \pmod{3k+2}}$ by $v_{i+2 \pmod{3k+2}}$ from D_i to get the neighbours of D_i in $ID_\gamma(C_{3k+2})$. When we replace $v_{i+1 \pmod{3k+2}}$ by $v_{i+2 \pmod{3k+2}}$, the vertex $v_{i+3 \pmod{3k+2}}$ is dominated by two dominating vertices, so we get the neighbour $D_{i+3 \pmod{3k+2}}$. Similarly, replacing $v_{i-1 \pmod{3k+2}}$ by $v_{i-2 \pmod{3k+2}}$ gives the neighbour $D_{i-3 \pmod{3k+2}}$. Hence D_i is adjacent to only $D_{i+3 \pmod{3k+2}}$ and $D_{i-3 \pmod{3k+2}}$. Then $D_0D_3 \cdots D_{3k}\, D_1D_4 \cdots D_{3k+1}\, D_2D_5 \cdots D_{3k-1}\, D_0$ is a cycle with $3k$+2 vertice in $ID_\gamma(C_{3k+2})$. This completes the proof.

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