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Full Paper

On starlikeness and uniform convexity of certain integral operators defined by Struve functions

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Abstract: This article deals with some functional inequalities involving generalised Struve functions, Struve functions and modified Struve functions. We aim to find the local univalence, starlikeness and uniform convexity of integral operators defined by generalised Struve function, Struve function and modified Struve functions.

Keywords: starlike functions, convex functions, Struve functions, integral operators

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INTRODUCTION

Let A denote the class of functions f which are analytic in the open unit disc $E=\{z:|z|<1\}$. These functions are of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n. \tag{1}$$

We denote by S, the set of all functions in A which are univalent in E. The subclasses S^* and C of S consist of all functions which map E on to a star-shaped domain with respect to the origin and convex domain respectively. The class UCV of uniformly convex function is defined as

$$UCV=\left\{f\in A:\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\left|\frac{zf''(z)}{f'(z)}\right|,\quad z\in E\right\}.$$

Let $K(\mu)$ denote the class of locally univalent and normalised analytic functions in E. These functions satisfy the condition

$$K(\mu)=\left\{f\in A:\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\frac{1}{2}-\mu,\quad -\frac{1}{2}<\mu\le\frac{1}{2},\ z\in E\right\}. \tag{2}$$

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The above condition is known as Ozaki's univalence condition, which is given by Ozaki [1].

Now we consider the second-order inhomogeneous differential equation

$$z^2 w''(z) + zw'(z) + (z^2 - l^2) w(z) = \frac{4\left(\frac{z}{2}\right)^{l+1}}{\sqrt{\pi}\,\Gamma\left(l+\frac{1}{2}\right)}. \tag{3}$$

The solution of the homogenous part is Bessel functions of order l, where l is a real or complex number. Bessel functions have been studied by several authors [2-8]. The particular solution of the inhomogeneous equation defined in (3) is called Struve function of order l, defined and studied by Struve [9]. It is defined as

$$X_l(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+l+1}}{\Gamma(n+3/2)\Gamma(n+l+3/2)}. \tag{4}$$

Now we consider the differential equation

$$z^2 w''(z) + zw'(z) - (z^2 + l^2) w(z) = \frac{4(z/2)^{l+1}}{\sqrt{\pi}\,\Gamma(l+1/2)}. \tag{5}$$

Equation (5) differs from equation (3) in the coefficients of w. Its particular solution is called the modified Struve function of order l. It is given as

$$Y_l(z) = -ie^{-ip\pi/2} X_l(iz) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+l+1}}{\Gamma(n+3/2)\,\Gamma(l+n+3/2)}. \tag{6}$$

Consider the second-order inhomogeneous equation

$$z^2 w''(z) + bzw'(z) + \left[cz^2 - l^2 + (1-b)l\right] w(z) = \frac{4(z/2)^{l+1}}{\sqrt{\pi}\,\Gamma(l+n+3/2)}, \tag{7}$$

where $b, c, l \in$X. This equation (7) generalises equations (3) and (5). In particular for $b=1$, $c=1$, we obtain (3) and for $b=1$, $c=-1$, we obtain (5). Its particular solution has the series form

$$w_{l,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n (z/2)^{2n+l+1}}{\Gamma(n+3/2)\,\Gamma(l+n+(b+2)/2)}. \tag{8}$$

This is called the generalised Struve function of order l. This series defined in (8) is convergent everywhere. We take the transformation

$$v_{l,b,c}(z) = 2^l \sqrt{\pi}\ \Gamma(l+b/2) z^{(-l-1)/2}\ w_{l,b,c}(\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-c/4)^n z^n}{(3/2)_n (q)_n}, \tag{9}$$

where $q = l + \frac{(b+2)}{2} \neq 0, -1, -2, \ldots$ and $(\gamma)_n = \frac{\Gamma(\gamma+n)}{\Gamma(n)} = \gamma(\gamma+1)\ldots(\gamma+n-1)$. The symbol $\Gamma(.)$ denotes the Gamma function. The function $v_{l,b,c}$ is analytic in the whole complex plane and satisfies the differential equation

$$4z^2 w''(z) + 2(2l+b+3)zw'(z) + \left[cz + 2l + b\right] w(z) = 2l + b.$$

This function unifies Struve functions and modified Struve functions. The function $v_{l,b,c}$ was introduced and studied by Orhan and Yugmur [10]. It was further investigated by Baricz et al. [11], Orhan and Yagmur [12] and Raza and Yagmur [13].

In the last few years, many mathematicians have set the univalence criteria of certain integral operators which preserve the class *S*. Several authors have studied the univalence criterion by using a variety of different analytic techniques, operators and special functions. Recently, Din et al. [14] studied the univalence of integral operators involving generalised Struve functions. These operators are defined as follows.

$$F_{l_i,b,c,\alpha_1,\ldots,\alpha_n,\beta}(z)=\left[\beta\int_0^z t^{\beta-1}\prod_{i=1}^{n}\left(\frac{v_{l_i,b,c}(t)}{t}\right)^{\frac{1}{\alpha_i}}dt\right]^{\frac{1}{\beta}}, \tag{10}$$

$$G_{v_{l_i,b,c,\gamma}}(z)=\left[(n\gamma+1)\int_0^z\prod_{i=1}^{n}\left\{v_{l_i,b,c}(t)\right\}^{\gamma}dt\right]^{\frac{1}{n\gamma+1}}, \tag{11}$$

$$H_{l,b,c,\lambda}(z)=\left[\lambda\int_0^z t^{\lambda-1}\left(e^{v_{l,b,c}(t)}\right)^{\lambda}dt\right]^{\frac{1}{\lambda}}. \tag{12}$$

In this paper our main aim is to study the starlikeness, convexity and univalence of the integral operators

$$F_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z) = \int_0^z\prod_{i=1}^{n}\left(\frac{v_{l_i,b,c}(t)}{t}\right)^{\alpha_i}dt,$$

$$W_{l_i,b,c,\delta_1,\ldots,\delta_n}(z) = \int_0^z\prod_{i=1}^{n}\left(v'_{l_i,b,c}(t)\right)^{\delta_i}dt.$$

Applications of Struve functions occur in water wave and surface wave problems, unsteady aerodynamics, resistive magneto hydro-dynamics instability theory and optical diffraction. More recently, Struve functions appeared in particle quantum dynamic studies of spin decoherence and nanotubes electrodynamics, potential theory and optics. These applications are explored in detail by Abramowitz and Stegun [15].

We need the following lemmas to prove our main results.

Lemma 1 [16]. If $g\in A$ satisfies

$$\left|1+\frac{zg''(z)}{g'(z)}\right|<2, \qquad \text{then } g\in S^*.$$

Lemma 2 [10]. If $g\in A$ satisfies

$$\left|\frac{zg''(z)}{g'(z)}\right|<\frac{1}{2}, \qquad \text{then } g\in UCV.$$

Now we give some functional inequalities which are very essential in proving our main results.

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Lemma 3 [10]. If the parameters $b, l \in \mathrm{P}$ and $c \in \mathrm{X}$, $q = l + \frac{b+2}{2}$ are so constrained that $q > \max\left\{0, \frac{7|c|}{24}\right\}$, then the function $v_{l,b,c} : E \to \mathrm{X}$ satisfies the following inequalities.

$$(i) \qquad \left|\frac{zv'_{l,b,c}(z)}{v_{l,b,c}(z)} - 1\right| \le \frac{|c|\left(6q-|c|\right)}{3\left(4q-|c|\right)\left(3q-|c|\right)}. \tag{13}$$

$$(ii) \qquad \left|\frac{zv''_{l,b,c}(z)}{v'_{l,b,c}(z)}\right| \le \frac{6|c|}{12q-7|c|}. \tag{14}$$

MAIN RESULTS

In this section we find the starlikeness and uniform convexity of the integral operators defined by generalised Struve functions. We shall use the above lemmas and inequalities to prove our results. We shall also use Ozaki's condition [1] for the univalence of these operators.

Starlikeness and Uniformly Convexity Criteria for Integral Operators

Theorem 1. Let $l_1, l_2, \ldots, l_n, b \in \mathrm{P}$, $c \in \mathrm{X}$ and $q_i > \frac{7|c|}{24}$ with $q_i = l_i + \frac{b+2}{2}$, $i = 1, \ldots, n$. Let the function $v_{l_i,b,c} : E \to \mathrm{X}$ be defined as

$$v_{l_i,b,c}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + b/2\right) z^{-l_i - 1/2}\, w_{l_i,b,c}\left(\sqrt{z}\right), \tag{15}$$

and suppose $q = \min\{q_1, q_2, \ldots, q_n\}$ and $\alpha_i \ (i = 1, 2, \ldots, n)$ are positive real numbers satisfying the relation

$$0 \le \frac{|c|\left(6q-|c|\right)}{3\left(4q-|c|\right)\left(3q-|c|\right)} \sum_{i=1}^{n} \alpha_i < 1. \tag{16}$$

Then the function $F_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z) \in S^*$.

Proof. We consider a function

$$F_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{v_{l_i,b,c}(t)}{t}\right)^{\alpha_i} dt. \tag{17}$$

Logarithmic differentiation of (17) and computations yield

$$\frac{zF''_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i z\left(\frac{v'_{l_i,b,c}(z)}{v_{l_i,b,c}(z)} - \frac{1}{z}\right).$$

Taking absolute value and by using triangle inequality, we have

$$\left|\frac{zF''_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}\right| = \sum_{i=1}^{n} \alpha_i \left|\frac{zv'_{l_i,b,c}(z)}{v_{l_i,b,c}(z)} - 1\right|.$$

By using Lemma 3 (i), we have

$$\left|\frac{zF''_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}\right|\le\sum_{i=1}^{n}\alpha_i\left(\frac{|c|(6q_i-|c|)}{3(4q_i-|c|)(3q_i-|c|)}\right).$$

For $z\in E$ and $q,q_i=l_i+\frac{b+2}{2}>\frac{7|c|}{24},\ \ \forall(i=1,2,...,n)$,we assume that the function

$$\tau:\left(\frac{7|c|}{24},\infty\right)\to\mathrm{P},$$

defined by

$$\tau(x)=\frac{|c|(6x-|c|)}{3(4x-|c|)(3x-|c|)},$$

is a decreasing function $\forall(i=1,2,...,n)$. Therefore,

$$\frac{|c|(6q_i-|c|)}{3(4q_i-|c|)(3q_i-|c|)}\le\frac{|c|(6q-|c|)}{3(4q-|c|)(3q-|c|)}.$$

Hence

$$\left|\frac{zF''_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z)}\right|\le\sum_{i=1}^{n}\alpha_i\left(\frac{|c|(6q-|c|)}{3(4q-|c|)(3q-|c|)}\right).$$

This implies that

$$\left|1+\frac{zF''_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z)}\right|\le\sum_{i=1}^{n}\alpha_i\left(\frac{|c|(6q-|c|)}{3(4q-|c|)(3q-|c|)}\right)+1.$$

Now by Lemma 1, we see that

$$\left|1+\frac{zF''_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z)}\right|<2$$

if (16) is satisfied. So by Lemma 1, the function $F_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z)\in S^*$.

Theorem 2. Let $l_i,b\in\mathrm{P}$, $c\in\mathrm{X}$ and $q_i>\frac{7|c|}{24}$ with $q_i=l_i+\frac{b+2}{2}$, $i=1,...,n$. Let the function $v_{l_i,b,c}:E\to\mathrm{X}$ be defined as

$$v_{l_i,b,c}(z)=2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i+{}^{(b+2)}\!/_{2}\right)z^{-l_i-1/2}\,w_{l_i,b,c}\left(\sqrt{z}\right).$$

Suppose $q=\min\{q_1,q_2,...,q_n\}$ and $\alpha_i\ (i=1,2,...,n)$ are positive real numbers satisfying the relation

$$0\le\frac{2|c|(6q-|c|)}{3(4q-|c|)(3q-|c|)}\sum_{i=1}^{n}\alpha_i<1. \tag{18}$$

Then the function $F_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z)\in UCV$.

Proof . Now as in Theorem 1, we take

$$\frac{zF''_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}=\sum_{i=1}^{n}\alpha_i\left(\frac{zv'_{l_i,b,c}(z)}{v_{l_i,b,c}(z)}-1\right).$$

Taking absolute value and by using triangle inequality, we have

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$$\left|\frac{zF''_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}\right| = \sum_{i=1}^{n}\alpha_i\left|\frac{zv'_{l_i,b,c}(z)}{v_{l_i,b,c}(z)}-1\right|.$$

By using the same method as in Theorem 1, we see that

$$\left|\frac{zF''_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}\right| \le \sum_{i=1}^{n}\alpha_i\left(\frac{|c|\left(6q-|c|\right)}{3\left(4q-|c|\right)\left(3q-|c|\right)}\right).$$

Now by Lemma 2, it follows that

$$\left|\frac{zF''_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}\right| < \frac{1}{2}$$

if (18) is satisfied. So by Lemma 2 the function $F_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z) \in UCV$.

In a similar manner as above, we can easily prove the following theorem.

Theorem 3. Let $l_i, b \in \mathrm{P}$, $c \in \mathrm{X}$ and $q_i > \frac{7|c|}{24}$ with $q_i = l_i + \frac{b+2}{2}$, $i = 1,\ldots,n$. Let the function $v_{l_i,b,c} : E \to \mathrm{X}$ be defined as

$$v_{l_i,b,c}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + {(b+2)}/{2}\right)z^{-l_i-1/2}\, w_{l_i,b,c}\left(\sqrt{z}\right).$$

Suppose $q = \min\{q_1, q_2, \ldots, q_n\}$ and δ_i $(i = 1,2,\ldots,n)$ are positive real numbers satisfying the relation

$$0 \le \frac{6|c|}{12q - 7|c|}\sum_{i=1}^{n}\delta_i < 1. \tag{19}$$

Then the function $W_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z) \in S^*$.

Struve Function

We obtain Struve function of the first kind of order l by setting $b = c = 1$ in (8), denoted by $X_l(z)$ and defined by (4). Let $\aleph_l : E \to \mathrm{X}$ be defined as

$$\aleph_l(z) = 2^l\sqrt{\pi}\,\Gamma\left(l + {3}/{2}\right)z^{(-l-1)/2}\, X_l\left(\sqrt{z}\right).$$

Notice that

$$\aleph_{\frac{-1}{2}}(z) = \sqrt{z}\sin\sqrt{z},$$

$$\aleph_{\frac{1}{2}}(z) = 2\left(1-\cos\sqrt{z}\right),$$

$$\aleph_{\frac{3}{2}}(z) = 4\left(1+\frac{2}{z}\right) - 8\left(\frac{\sin\left(\sqrt{z}\right)}{\sqrt{z}} + \frac{\cos\left(\sqrt{z}\right)}{\sqrt{z}}\right).$$

Corollary 1. (I) Consider the function $\aleph_{l_i} : E \to \mathrm{X}$ defined by

$$\aleph_{l_i}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + {3}/{2}\right)z^{(-l_i-1)/2}\, X_{l_i}\left(\sqrt{z}\right).$$

Let $l_1, l_2, \ldots, l_n > -1.75$ and $l = \min\{l_1, l_2, \ldots, l_n\}$. Also, let the parameters $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers satisfying the relation

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$$0 \le \frac{4(3l+4)}{3(24l^2+58l+35)} \sum_{i=1}^{n} \alpha_i < 1. \qquad (20)$$

Then the integral operator defined by

$$F_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{\aleph_{l_i}(t)}{t} \right)^{\alpha_i} dt$$

is in class S^*. In particular, the function $F_{\frac{3}{2},\alpha} : E \to \mathrm{X}$ defined by

$$F_{\frac{3}{2},\alpha}(z) = \int_0^z \left(\frac{1}{t} \aleph_{\frac{3}{2}}(t) \right)^{\alpha} dt = \int_0^z \left(\frac{4}{t}\left(1+\frac{2}{t}\right) - \frac{8}{t}\left(\frac{\sin\left(\sqrt{t}\right)}{\sqrt{t}} + \frac{\cos\left(\sqrt{t}\right)}{\sqrt{t}} \right) \right)^{\alpha} dt \qquad (21)$$

is in class S^* for $\alpha \le \frac{264}{17}$. The function $F_{\frac{1}{2},\alpha} : E \to \mathrm{X}$ defined by

$$F_{\frac{1}{2},\alpha}(z) = \int_0^z \left(\frac{1}{t} \aleph_{\frac{1}{2}}(t) \right)^{\alpha} dt = \int_0^z \left(\frac{2}{t}\left(1-\cos\sqrt{t}\right) \right)^{\alpha} dt \qquad (22)$$

is in class S^* for $\alpha \le \frac{105}{11}$. The function $F_{\frac{-1}{2},\alpha} : E \to \mathrm{X}$ defined by

$$F_{-\frac{1}{2},\alpha}(z) = \int_0^z \left(\frac{1}{t} \aleph_{\frac{-1}{2}}(t) \right)^{\alpha} dt = \int_0^z \left(\frac{\sin\sqrt{t}}{\sqrt{t}} \right)^{\alpha} dt \qquad (23)$$

is in class S^* for $\alpha \le \frac{18}{5}$.

(II) Let $l_1, l_2, \ldots, l_n > -1.75$ and consider the function $\aleph_{l_i} : E \to \mathrm{X}$ defined by

$$\aleph_{l_i}(z) = 2^{l_i} \sqrt{\pi}\, \Gamma\left(l_i + 3/2\right) z^{(-l_i-1)/2} X_{l_i}\left(\sqrt{z}\right)$$

with $l = \min\{l_1, l_2, \ldots, l_n\}$. Also, let the parameter $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers satisfying the relation

$$0 \le \frac{8(3l+4)}{3(24l^2+58l+35)} \sum_{i=1}^{n} \alpha_i < 1. \qquad (24)$$

Then the integral operator defined by

$$F_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{1}{t} \aleph_{l_i}(t) \right)^{\alpha_i} dt$$

is in class UCV. In particular, the function $F_{\frac{3}{2},\alpha} : E \to \mathrm{X}$ defined by (21) is in class UCV for $\alpha \le \frac{264}{34}$. The function $F_{\frac{1}{2},\alpha} : E \to \mathrm{X}$ defined by (22) is in class UCV for $\alpha \le \frac{105}{22}$. The function $F_{\frac{-1}{2},\alpha} : E \to \mathrm{X}$ defined by (23) is in class UCV for $\alpha \le \frac{18}{10}$. We found the values of α by giving $l = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$ respectively in (24).

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Corollary 2. Consider the function $\aleph_{l_i} : E \to X$ defined by

$$\aleph_{l_i}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + \tfrac{3}{2}\right) z^{(-l_i-1)/2} X_{l_i}\left(\sqrt{z}\right).$$

(I) Let $l_1, l_2, ..., l_n > -1.75$ and $l = \min\{l_1, l_2, ..., l_n\}$. Also, let the parameters $\delta_1, \delta_2, ..., \delta_n$ be positive real numbers satisfying the relation

$$0 \le \frac{6}{12l + 11}\sum_{i=1}^{n}\delta_i < 1.$$

Then the integral operator defined by

$$W_{l_i,\delta_1,...,\delta_n}(z) = \int_0^z \prod_{i=1}^{n}\left(\aleph'_{l_i}(t)\right)^{\delta_i} dt$$

is in class S^*.

(II) Let $l_1, l_2, ..., l_n > -1.75$ and consider the function $\aleph_{l_i} : E \to X$ defined by

$$\aleph_{l_i}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + \tfrac{3}{2}\right) z^{(-l_i-1)/2} X_{l_i}\left(\sqrt{z}\right)$$

with $l = \min\{l_1, l_2, ..., l_n\}$. Also, let the parameters $\delta_1, \delta_2, ..., \delta_n$ be positive real numbers satisfying the relation

$$0 \le \frac{12}{12l + 11}\sum_{i=1}^{n}\delta_i < 1.$$

Then the integral operator defined by

$$W_{l_i,\delta_1,...,\delta_n}(z) = \int_0^z \prod_{i=1}^{n}\left(\aleph'_{l_i}(t)\right)^{\delta_i} dt$$

is in class UCV.

Modified Struve Function

We obtain the modified Struve function by setting $b = 1,\ c = -1$ in (8), denoted by $Y_l(z)$ and defined by (6). Let $\Upsilon_{l_i} : E \to X$ be defined as

$$\Upsilon_{l_i}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + \tfrac{3}{2}\right) z^{(-l_i-1)/2} Y_{l_i}\left(\sqrt{z}\right).$$

We see that

$$\Upsilon_{\frac{1}{2}}(z) = 2\left(\cos\sqrt{z} - 1\right).$$

Corollary 3. Consider the function $\Upsilon_{l_i} : E \to X$ defined as

$$\Upsilon_{l_i}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + \tfrac{3}{2}\right) z^{(-l_i-1)/2} Y_{l_i}\left(\sqrt{z}\right).$$

(I) Let $l_1, l_2, ..., l_n > -1.75$ and $l = \min\{l_1, l_2, ..., l_n\}$. Also, let the parameters $\alpha_1, \alpha_2, ..., \alpha_n$ be positive real numbers satisfying the relation

$$0 \le \frac{4(3l + 4)}{3\left(24l^2 + 58l + 35\right)}\sum_{i=1}^{n}\alpha_i < 1.$$

Then the integral operator defined by

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$$F_{l_i,\alpha_1,\ldots,\alpha_n}(z)=\int_0^z\prod_{i=1}^n\left(\frac{\Upsilon_{l_i}(t)}{t}\right)^{\alpha_i}dt$$

is in class S^*. In particular, the function $F_{\frac{1}{2},\alpha}:E\to \mathrm{X}$ defined by

$$F_{\frac{1}{2},\alpha}(z)=\int_0^z\left(\frac{2}{t}\left(1-\cos\sqrt{t}\right)\right)^{\alpha}dt$$

is in class S^* for $\alpha\le\frac{264}{17}$.

(II) Let $l_1,l_2,\ldots,l_n>-1.75$ and $l=\min\{l_1,l_2,\ldots,l_n\}$. Also, let the parameters $\alpha_1,\alpha_2,\ldots,\alpha_n$ be positive real numbers satisfying the relation

$$0\le\frac{8(3l+4)}{3\left(24l^2+58l+35\right)}\sum_{i=1}^n\alpha_i<1.$$

Then the integral operator defined by

$$F_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z)=\int_0^z\prod_{i=1}^n\left(\frac{\Upsilon_{l_i}(t)}{t}\right)^{\alpha_i}dt$$

is in class UCV. In particular, the function $F_{\frac{1}{2},\alpha}:E\to \mathrm{X}$ defined by

$$F_{\frac{1}{2},\alpha}(z)=\int_0^z\left(\frac{2}{t}\left(1-\cos\sqrt{t}\right)\right)^{\alpha}dt$$

is in class UCV for $\alpha\le\frac{105}{44}$.

Corollary 4. Consider the function $\Upsilon_{l_i}:E\to \mathrm{X}$ defined by

$$\Upsilon_{l_i}(z)=2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i+\tfrac{3}{2}\right)z^{(-l_i-1)/2}\,Y_{l_i}\left(\sqrt{z}\right).$$

(I) Let $l_1,l_2,\ldots,l_n>-1.75$ and $l=\min\{l_1,l_2,\ldots,l_n\}$. Also, let the parameter $\delta_1,\delta_2,\ldots,\delta_n$ be positive real numbers satisfying the relation

$$0\le\frac{6}{12l+11}\sum_{i=1}^n\delta_i<1.$$

Then the integral operator defined by

$$W_{l_i,\delta_1,\ldots,\delta_n}(z)=\int_0^z\prod_{i=1}^n\left(\Upsilon'_{l_i}(t)\right)^{\delta_i}dt$$

is in class S^*.

(II) Let $l_1,l_2,\ldots,l_n>-1.75$ and $l=\min\{l_1,l_2,\ldots,l_n\}$. Also, let the parameters $\delta_1,\delta_2,\ldots,\delta_n$ be positive real numbers satisfying the relation

$$0\le\frac{6}{12l+11}\sum_{i=1}^n\delta_i<1.$$

Then the integral operator defined by

$$W_{l_i,\delta_1,\ldots,\delta_n}(z)=\int_0^z\prod_{i=1}^n\left(\Upsilon'_{l_i}(t)\right)^{\delta_i}dt$$

is in class UCV.

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Local Univalence Criteria for Integral Operator

Now we prove the local univalence criteria for integral operators by using Ozaki's condition defined by (2).

Theorem 4. Let $l_1, l_2, ..., l_n, b \in \mathrm{P}$, $c \in \mathrm{X}$ and $q_i > \frac{7|c|}{24}$ with $q_i = l_i + \frac{b+2}{2}$, $i = 1, ..., n$. Let the function $v_{l_i,b,c} : E \to \mathrm{X}$ be defined as

$$v_{l_i,b,c}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + {b+2}/{2}\right) z^{-l_i - 1/2}\, w_{l_i,b,c}\left(\sqrt{z}\right),$$

and suppose $q = \min\{q_1, q_2, ..., q_n\}$ and $\alpha_i\,(i = 1, 2, ..., n)$ are positive real numbers satisfying the relation

$$0 \le \frac{|c|(6q - |c|)}{3(4q - |c|)(3q - |c|)} \sum_{i=1}^{n} \alpha_i < 1.$$

Then the function $F_{l_i,b,c,\alpha_1,...,\alpha_n} \in K(\mu)$.

Proof . We consider a function

$$F_{l_i,b,c,\alpha_1,...,\alpha_n}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{v_{l_i,b,c}(t)}{t} \right)^{\alpha_i} dt. \tag{25}$$

Logarithmic differentiation of (25) and simple computations yield

$$\begin{aligned} \frac{zF''_{l_i,b,c,\alpha_1,...\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,...\alpha_n}(z)} &= \sum_{i=1}^{n} \alpha_i z \left(\frac{v'_{l_i,b,c}(z)}{v_{l_i,b,c}(z)} - \frac{1}{z} \right) \\ &= \sum_{i=1}^{n} \alpha_i \left(\frac{zv'_{l_i,b,c}(z)}{v_{l_i,b,c}(z)} - 1 \right) \\ &= \sum_{i=1}^{n} \alpha_i \left(\frac{zv'_{l_i,b,c}(z)}{v_{l_i,b,c}(z)} \right) - \sum_{i=1}^{n} \alpha_i. \end{aligned}$$

This implies that

$$1 + \frac{zF''_{l_i,b,c,\alpha_1,...\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,...\alpha_n}(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zv'_{l_i,b,c}(z)}{v_{l_i,b,c}(z)} \right) - \sum_{i=1}^{n} \alpha_i + 1. \tag{26}$$

By taking the real value of (26), we see that

$$\Re\left(1 + \frac{zF''_{l_i,b,c,\alpha_1,...\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,...\alpha_n}(z)} \right) = \sum_{i=1}^{n} \alpha_i \Re\left(\frac{zv'_{l_i,b,c}(z)}{v_{l_i,b,c}(z)} \right) + \left(1 - \sum_{i=1}^{n} \alpha_i \right). \tag{27}$$

Now consider the inequality

$$\left| \frac{zv'_{l,b,c}(z)}{v_{l,b,c}(z)} - 1 \right| \le \frac{|c|(6q - |c|)}{3(4q - |c|)(3q - |c|)}.$$

This implies that

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$$\Re\left(\frac{zv'_{l_i,b,c}(z)}{v_{l_i,b,c}(z)}\right)>1-\frac{|c|\left(6q_i-|c|\right)}{3\left(4q_i-|c|\right)\left(3q_i-|c|\right)}. \tag{28}$$

Now by using (28) in (27), we get

$$\Re\left(1+\frac{zF''_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}\right)>\sum_{i=1}^{n}\alpha_i\left(1-\frac{|c|\left(6q_i-|c|\right)}{3\left(4q_i-|c|\right)\left(3q_i-|c|\right)}\right)+\left(1-\sum_{i=1}^{n}\alpha_i\right)$$

$$>\sum_{i=1}^{n}\alpha_i-\sum_{i=1}^{n}\alpha_i\left(\frac{|c|\left(6q_i-|c|\right)}{3\left(4q_i-|c|\right)\left(3q_i-|c|\right)}\right)-\sum_{i=1}^{n}\alpha_i+1$$

$$>1-\sum_{i=1}^{n}\alpha_i\left(\frac{|c|\left(6q_i-|c|\right)}{3\left(4q_i-|c|\right)\left(3q_i-|c|\right)}\right).$$

For $z\in E$ and $q,q_i=l_i+\frac{b+2}{2}>\frac{7|c|}{24},\ \forall\left(i=1,2,\ldots,n\right)$, we assume that the function $\tau:\left(\frac{7|c|}{24},\infty\right)\to \mathrm{P}$ defined by

$$\tau(x)=\frac{|c|\left(6x-|c|\right)}{3\left(4x-|c|\right)\left(3x-|c|\right)}$$

is a decreasing function $\forall\left(i=1,2,\ldots,n\right)$. Therefore,

$$\frac{|c|\left(6q_i-|c|\right)}{3\left(4q_i-|c|\right)\left(3q_i-|c|\right)}\leq\frac{|c|\left(6q-|c|\right)}{3\left(4q-|c|\right)\left(3q-|c|\right)}.$$

Hence

$$\Re\left(1+\frac{zF''_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}{F'_{l_i,b,c,\alpha_1,\ldots\alpha_n}(z)}\right)>1-\sum_{i=1}^{n}\alpha_i\left(\frac{|c|\left(6q-|c|\right)}{3\left(4q-|c|\right)\left(3q-|c|\right)}\right)$$

$$>\frac{1}{2}+\frac{1}{2}-\sum_{i=1}^{n}\alpha_i\left(\frac{|c|\left(6q-|c|\right)}{3\left(4q-|c|\right)\left(3q-|c|\right)}\right)$$

$$>\frac{1}{2}-\left\{\sum_{i=1}^{n}\alpha_i\left(\frac{|c|\left(6q-|c|\right)}{3\left(4q-|c|\right)\left(3q-|c|\right)}\right)-\frac{1}{2}\right\}$$

$$>\frac{1}{2}-\mu,$$

where

$$\mu=\sum_{i=1}^{n}\alpha_i\left(\frac{|c|\left(6q-|c|\right)}{3\left(4q-|c|\right)\left(3q-|c|\right)}\right)-\frac{1}{2}.$$

So $F_{l_i,b,c,\alpha_1,\ldots,\alpha_n}(z)\in K(\mu)$ by (2).

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Theorem 5. Let $l_1, l_2, ..., l_n, b \in \mathrm{P}$, $c \in \mathrm{X}$ and $q_i > \frac{7|c|}{24}$ with $q_i = l_i + \frac{b+2}{2}$, $i = 1, ..., n$. Let the function $v_{l_i,b,c} : E \to \mathrm{X}$ be defined as

$$v_{l_i,b,c}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + {(b+2)}/{2}\right) z^{-l_i - 1/2}\, w_{l_i,b,c}\left(\sqrt{z}\right).$$

Suppose $q = \min\{q_1, q_2, ..., q_n\}$ and $\delta_i\,(i = 1, 2, ..., n)$ are positive real numbers satisfying the relation

$$0 \le \frac{6|c|}{12q - |c|}\sum_{i=1}^{n}\delta_i < 1.$$

Then the function $W_{l_i,b,c,\delta_1,...,\delta_n}(z) \in K(\mu)$.

Proof. The proof is similar to that of Theorem 4.

Corollary 5. Consider the function $\aleph_{l_i} : E \to \mathrm{X}$ defined by

$$\aleph_{l_i}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + 3/2\right) z^{(-l_i-1)/2}\, X_{l_i}\left(\sqrt{z}\right).$$

Let $l_1, l_2, ..., l_n > -1.75$ and $l = \min\{l_1, l_2, ..., l_n\}$. Also, let the parameters $\alpha_1, \alpha_2, ..., \alpha_n$ be positive real numbers satisfying the relation

$$0 \le \frac{4(3l+4)}{3\left(24l^2 + 58l + 35\right)}\sum_{i=1}^{n}\alpha_i < 1.$$

Then the integral operator defined by

$$F_{l_i,\alpha_1,...,\alpha_n}(z) = \int_0^z \prod_{i=1}^{n}\left(\frac{\aleph_{l_i}(t)}{t}\right)^{\alpha_i} dt \in K(\mu).$$

Corollary 6. Consider the function $\aleph_{l_i} : E \to \mathrm{X}$ defined by

$$\aleph_{l_i}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + 3/2\right) z^{(-l_i-1)/2}\, X_{l_i}\left(\sqrt{z}\right).$$

Let $l_1, l_2, ..., l_n > -1.75$ and $l = \min\{l_1, l_2, ..., l_n\}$. Also, let the parameters $\delta_1, \delta_2, ..., \delta_n$ be positive real numbers satisfying the relation

$$0 \le \frac{6}{12l + 11}\sum_{i=1}^{n}\delta_i < 1.$$

Then the integral operator defined by

$$W_{l_i,\delta_1,...,\delta_n}(z) = \int_0^z \prod_{i=1}^{n}\left(\aleph'_{l_i}(t)\right)^{\delta_i} dt \in K(\mu).$$

Corollary 7. Consider the function $\Upsilon_{l_i} : E \to \mathrm{X}$ defined as

$$\Upsilon_{l_i}(z) = 2^{l_i}\sqrt{\pi}\,\Gamma\left(l_i + 3/2\right) z^{(-l_i-1)/2}\, Y_{l_i}\left(\sqrt{z}\right).$$

Let $l_1, l_2, ..., l_n > -1.75$ and $l = \min\{l_1, l_2, ..., l_n\}$. Also, let the parameters $\alpha_1, \alpha_2, ..., \alpha_n$ be positive real numbers satisfying the relation

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$$0 \le \frac{4(3l+4)}{3(24l^2+58l+35)} \sum_{i=1}^{n} \alpha_i < 1.$$

Then the integral operator defined by

$$F_{l_i,\alpha_1,...,\alpha_n,\beta}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{\Upsilon_{l_i}(t)}{t} \right)^{\alpha_i} dt \in K(\mu).$$

Corollary 8. Consider the function $\Upsilon_{l_i} : E \to \mathrm{X}$ defined by

$$\Upsilon_{l_i}(z) = 2^{l_i} \sqrt{\pi}\, \Gamma\left(l_i + 3/2\right) z^{(-l_i-1)/2}\, Y_{l_i}\left(\sqrt{z}\right).$$

Let $l_1, l_2, ..., l_n > -1.75$ and $l = \min\{l_1, l_2, ..., l_n\}$. Also, let the parameters $\delta_1, \delta_2, ..., \delta_n$ be positive real numbers satisfying the relation

$$0 \le \frac{6}{12l+11} \sum_{i=1}^{n} \delta_i < 1.$$

Then the integral operator defined by

$$W_{l_i,\delta_1,...,\delta_n,\beta}(z) = \int_0^z \prod_{i=1}^{n} \left(\Upsilon'_{l_i}(t) \right)^{\delta_i} dt \in K(\mu).$$

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