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Full Paper

Presentation of inverse LA-semigroups

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Abstract: In this paper we attempt to lay a foundation to define an inverse LA-semigroup by generators and relations. We determine a number of presentations of inverse LA-semigroup, LA-group and commutative inverse LA-semigroup. Furthermore, we devise direct product and wreath product of inverse LA-semigroups to analyse the structure of finite inverse LA-semigroups effectively.

Keywords: LA-semigroup, inverse LA-semigroup, LA-group, direct product, wreath product

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INTRODUCTION

The left almost semigroup (LA-semigroup) is a groupoid together with the left invertive law $(uv)w = (wv)u$. Similarly, the right almost semigroup is a groupoid together with the right invertive law $u(vw) = w(vu)$. Kazim and Naseeruddin [1] introduced this algebraic structure in the early 1970s. Later, Mushtaq and Yusuf [2] introduced some new ideas related to LA-semigroups. Several names such as right modular groupoids, left invertive groupoids or AG-groupoids were also used in place of LA-semigroups [3-5]. LA-semigroups are non-associative, but they turn into commutative semigroups under certain conditions. For instance, an LA-semigroup containing the right identity becomes a commutative semigroup.

An LA-semigroup is always medial, but its converse is not true. Younas and Mushtaq [6] provided an example to substantiate that the set $A = \{0,1,2,3,4\}$ in Table 1 satisfies the medial identity without being an LA-semigroup. Here, $(0 * 3) * 1 \neq (1 * 3) * 0$ substantiates it.

Table 1. A medial groupoid which does not satisfy the left invertive law

$*$	0	1	2	3	4
0	4	4	4	2	4
1	4	0	0	4	4
2	4	0	4	2	4
3	0	2	2	0	0
4	4	4	4	4	4

In the following we use $\boldsymbol{L}$ to denote an LA-semigroup. Furthermore, we use $\boldsymbol{\mathcal{J}}$ ={**€:** € is an inverse LA-semigroup} to represent the class containing all inverse LA-semigroups and $\boldsymbol{\mathcal{L}}$ ={**£:** £ is a left permutable inverse LA-semigroup} to represent the class containing all left permutable inverse LA-semigroups. The cardinality is the size or order of an LA-semigroup (inverse LA-semigroup). The period of an element $u \in \boldsymbol{L}$ is the least positive integer n such that $u^{n+1} = u$.

Mushtaq and Yusuf [2] introduced the identity $(uu)u = u(uu)$ to define the associative powers of the elements in an LA-semigroup. An LA-semigroup with this additional property is called locally associative LA-semigroup. Later, Mushtaq and Iqbal [7] investigated some decompositions of locally associative LA-semigroups.

Mushtaq and Khan [8-9] proved several interesting facts of an LA-semigroup using the weak associative law $(uv)w = v(uw)$. One of these results is that the identities $(uv)w = v(uw)$ and $(uv)w = v(wu)$ are equivalent in an LA-semigroup. Consequently, the identity $u(vw) = u(wv)$ also holds. Distler et al. [10] examined that the smallest LA-semigroup satisfying the weak associative law is of order 6 which are just nine in number.

An LA-semigroup $\boldsymbol{L}$ satisfying the left permutable law, i.e. $u(vw) = v(uw)$ for every $u, v, w \in \boldsymbol{L}$, is a left permutable LA-semigroup. Protić and Božinović [11] proved that in a left permutable LA-semigroup the paramedial identity holds naturally, i.e. $(uv)(wx) = (xv)(wu)$. It is easy to see that a paramedial left permutable groupoid may not be an LA-semigroup. For instance, Table $2(i)$ represents a left permutable groupoid which contains a left identity without being an LA-semigroup, and Table $2(ii)$ represents a paramedial left permutable groupoid without being an LA-semigroup.

Table 2. Left permutable groupoids which are not LA-semigroups

(i)

	0	1	2	3	4
0	4	4	2	4	2
1	4	4	2	2	2
2	2	2	2	2	2
3	0	1	2	3	4
4	2	2	2	4	2

(ii)

	0	1	2	3	4
0	4	4	4	2	4
1	4	4	4	0	4
2	4	4	4	2	4
3	4	0	4	1	4
4	4	4	4	4	4

An inverse LA-semigroup is an LA-semigroup $\boldsymbol{L}$ which has $u' \in \boldsymbol{L}$ for every $u \in \boldsymbol{L}$ so that $(uu')u = u$ and $(u'u)u' = u'$. Then u' is called the inverse of u. Let $V(u)$ be the set of all inverses of u. Mushtaq and Iqbal [12] introduced the concept of an inverse LA-semigroup. They also investigated some structural properties of an inverse LA-semigroup satisfying the weak associative law.

Younas [13] generated substitution boxes by using a presentation discussed in this article. Furthermore, he applied a substitution box to create confusion in an image encryption scheme along with discrete and continuous chaotic systems to complement the diffusion characteristics. Based on performance analysis, he concluded that the proposed algorithm has better information security properties and better performance.

This article provides a foundation for the concept of defining an inverse LA-semigroup by generators and relations. Here, we describe some basic definitions and facts concerning inverse LA-semigroups. We devise a method for presenting a Cayley table of the inverse LA-semigroup by generators and relations. We obtain the presentations of a few inverse LA-semigroups and generalise some of these forms of presentation, which lays a foundation for the study of inverse LA-semigroups by Cayley graphs. In the end we show that the direct product and wreath product of the inverse LA-semigroup by an inverse LA-monoid again follow the inverse LA-semigroups containing a left identity.

PRELIMINARIES

An element e of an LA-semigroup $\boldsymbol{L}$ is the left identity (right identity) of $\boldsymbol{L}$, if $eu = u$, $(ae = u)$ for all $u \in \boldsymbol{L}$. An LA-semigroup containing a left identity is an LA-monoid. An element $u \in \boldsymbol{L}$ has left inverse if there exists an element $u^{-1} \in \boldsymbol{L}$ such that $u^{-1}u = e$. An LA-group is an LA-monoid in which every element has its left inverse.

Dudek and Gigon [14] investigated certain properties of left permutable inverse LA-semigroups, i.e. inverse LA-semigroup satisfying the identity $u(vw) = v(uw)$. It is easy to observe that every LA-group is a left permutable inverse LA-semigroup but not conversely. It was also proved that a left permutable inverse LA-semigroup £ with $u'u = uu'$ for all $u \in$ £ is precisely an LA-group.

Younas and Mushtaq [6] used a natural partial order to show that the set of idempotent elements in a left permutable inverse LA-semigroup is an order ideal. Furthermore, they used the compatibility relations to ascertain important characteristics of meets and joins in the left permutable inverse LA-semigroups. In the end, they furnished the conditions under which a left permutable inverse LA-semigroup is infinitely distributive.

Mushtaq and Iqbal [12] investigated some basic characteristics of inverses in an inverse LA-semigroup given in (i) to (v) of Proposition 1. In (vi) of Proposition 1, Božinović et al. [15] furnished a condition for the existence of idempotent elements in € $\in \boldsymbol{\mathcal{J}}$.

Proposition 1. Let € $\in \boldsymbol{\mathcal{J}}$. Then

(i) $(a')' = a$ for all $a \in$ €.

(ii) If e is an idempotent element of €, then $e' = e$.

(iii) $\left(\left(\left((a_1a_2)a_3\right)\dots\right)a_n\right)' = \left(\left(\left((a_1'a_2')a_3'\right)\dots\right)a_n'\right)$ for all $a_1, a_2, \dots, a_n \in$ €, $n \geq 2$.

(iv) $(ab)^{\mathrm{n}} = a^{\mathrm{n}}b^{\mathrm{n}}$ for all $a, b \in$ € and $n \in \mathbb{Z}$.

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(v) $(a^m)^n = a^{mn}$ for all $a \in$ € and $m, n \in \mathbb{Z}$.
(vi) If $aa' = a'a$, then aa' and $a'a$ are idempotent elements for all $a \in$ €.
(vii) The powers of an element $a \in$ € having period m are defined as $a^m = a^{m-1}a = (a^{m-2}a)a = \big((a^{m-3}a)a\big)a = \cdots = (((\big((aa)a)a\big)\cdots)a)a$ m-times. Furthermore, $a^i a = a^{i+1} \neq aa^i$ for $1 \leq i \leq m$.

Remark 1. For any £ $\in \boldsymbol{\mathcal{L}}$ and $a \in$ £, $aa' = a'a$ if and only if aa' and $a'a$ are idempotent elements of £.

The above remark is a consequence of part (vi) of Proposition 1 for any £ $\in \boldsymbol{\mathcal{L}}$. The following proposition provides a connection between a left permutable inverse LA-semigroup and an LA-group.

Proposition 2. Every £ $\in \boldsymbol{\mathcal{L}}$, in which $aa' = a'a$ for all $a \in$ £ with a unique idempotent element, is precisely an LA-group.

PRESENTATION OF INVERSE LA-SEMIGROUPS

The presentation of an inverse LA-semigroup is of vital importance just as it is for groups and semigroups. If one is not careful enough to distinguish between the elements of an LA-semigroup and words that describe these elements, utter confusion is likely to ensue. This paper marks the first attempt of describing a technique of finding the finite presentations of inverse LA-semigroups, which is still far from being comprehensive. This technique enables us to study the finite inverse LA-semigroups in a more efficacious way. It also helps us to further investigate the inverse LA-semigroups graphically just like groups and semigroups.

The main problem which arises here is that of recognising when two sets of generators and relations represent the same LA-semigroup. LA-semigroups can also be described by exhibiting its Cayley table just like groups and semigroups. Of course, the use of a Cayley table is not possible for infinite LA-semigroups, nor even feasible for a finite LA-semigroup of large order. Furthermore, the Cayley table contains redundant information, so that it is not an efficient device. For instance, one can reduce the necessary information to determine all the elements of the LA-semigroup presented in Table 3. We note that the elements s and v along with the relations $s^4 = s$ and $vs = v = ss^2$ generate it. It leads to the method of defining an LA-semigroup by generators and relations.

Table 3. An inverse LA-semigroup of order 4

	s	t	u	v
s	t	v	u	s
t	u	s	t	v
u	s	u	v	t
v	v	t	s	u

A word over the LA-semigroup $\boldsymbol{L}$ is a finite string $w = x_1x_2x_3 \ldots x_l = ((x_1x_2)x_3\ldots)x_l$ with each $x_i \in \boldsymbol{L}$. The length of w is denoted by $l = l(w) = |w|$. An empty word $\in$ is a word

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with $l = 0$. According to Neumann [16], the semigroup presentations are usually expressed in the form

$$\Pi =< x_1, \ldots, x_n \mid s_1 = r_1, \ldots, s_m = r_m >,$$

where $m, n \in \mathbb{N}$ and $s_i = r_i, i = 1, \ldots, m$ are relations in terms of generators $x_1, \ldots, x_n$. Campbell et al. [17-19] used the symbols SGP(Π) and GP(Π) to distinguish between the semigroup and group defined by Π. Also, they investigated necessary and sufficient conditions for the minimum two-sided ideals of SGP(Π) to form a disjoint union of copies of the group GP(Π). We use the notations ILAS(Π), LPILAS(Π), ILAM(Π) and LAGP(Π) to distinguish the presentations of inverse LA-semigroup, left permutable inverse LA-semigroup, inverse LA-monoid and inverse LA-group respectively.

First, we construct some examples to establish a foundation for more general and composite structures. Later, we use these concepts to elaborate on the products (direct and wreath) in the next section.

Example 1. The presentation $\text{ILAS}(\Pi) =< a, b \mid aa^2 = b = ba, a^4 = a >$ defines an inverse LA-semigroup of order 4 with the left inverses $a' = a^3 \text{ and } b' = a^2$ (Table 4).

Table 4. Inverse LA-semigroup having two generators

	a	a^2	a^3	b
a	a^2	b	a^3	a
a^2	a^3	a	a^2	b'
a^3	a	a^3	b	a^2
b	b	a^2	a	a^3

Example 2. The presentation $\text{ILAM}(\Pi) =< a \mid a^5 = a = aa^2, aa^3 = a^4 >$ defines a cyclic inverse LA-monoid of order 4 with the left inverse of $a' = a^3$, and a^4 is the left identity (Table 5).

Table 5. Inverse LA-monoid of order 4

	a	a^2	a^3	a^4
a	a^2	a	a^4	a^3
a^2	a^3	a^4	a	a^2
a^3	a^4	a^3	a^2	a
a^4	a	a^2	a^3	a^4

The succeeding theorem provides a general presentation for the cyclic inverse LA-group of order n.

Theorem 1. The presentation

$$\text{LAGP}(\Pi) = < a \mid a^{n+1} = a = aa^{\frac{n}{2}}, a^m a^{n-m} = a^n, 1 \leq m \leq n-1, n = 4k, m, k \in \mathbb{Z} >$$

defines an LA-group.

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Proof. Since a is the only generator with period n, i.e. $a^{n+1} = a$, therefore

$$a^m = \left(\left(((aa)a)\ldots\right)a\right) = \left(((a^{n+1}a)a)\ldots\right)a = ((a^{n+2}a)\ldots)a = \ldots = a^{n+m}.$$

Furthermore, by the variation of m, it is clear that $a^m a^{n-m} = a^n = a^{n-m}a^m$, and $a^m a^{n-m}$ is the idempotent element in LAGP(Π), i.e.

$$(a^m a^{n-m})^2 = (a^m a^{n-m})(a^m a^{n-m}) = (a^m a^{n-m})(a^{n-m}a^m)$$
$$= \left((a^{n-m}a^m)a^{n-m}\right)a^m = a^{n-m}a^m = a^m a^{n-m}.$$

Note that $a^n a^m = (a^m a^{n-m})a^m = a^m$ for each m and n where $1 \leq m \leq n-1$. This proves that a^n is the left identity and unique idempotent of LAGP(Π). Thus, by Proposition 2 and Remark 1, LAGP(Π) represents an inverse LA-group. □

The next example shows a finite presentation of the cyclic inverse LA-semigroup defined by Table 6 in detail.

Example 3. If $n = 8$, then LAGP(Π) $= < a \mid a^9 = a = aa^4, a^m a^{8-m} = a^8, 1 \leq m \leq 7 >$. Note that $aa^7 = a^8, a^2a^6 = a^8, a^3a^5 = a^8, a^4a^4 = a^8, a^5a^3 = a^8, a^6a^2 = a^8, a^7a = a^8$. The following Cayley table (Table 6) explicates the structure of cyclic inverse LA-semigroup.

Table 6. Cyclic inverse LA-semigroup

	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8
a	a^2	a^7	a^4	a	a^6	a^3	a^8	a^5
a^2	a^3	a^4	a^5	a^6	a^7	a^8	a	a^2
a^3	a^4	a	a^6	a^3	a^8	a^5	a^2	a^7
a^4	a^5	a^6	a^7	a^8	a	a^2	a^3	a^4
a^5	a^6	a^3	a^8	a^5	a^2	a^7	a^4	a
a^6	a^7	a^8	a	a^2	a^3	a^4	a^5	a^6
a^7	a^8	a^5	a^2	a^7	a^4	a	a^6	a^3
a^8	a	a^2	a^3	a^4	a^5	a^6	a^7	a^8

We determine the inverse LA-semigroups of orders 5, 9, 11, 19 and 25 belonging to a specific class. All these inverse LA-semigroups have two generators and common relations $aa^{n-2} = b = ba$ and $a^n = a$, where n denotes the order of the inverse LA-semigroup (Table 7). There are no other such inverse LA-semigroups between the orders 5, 9, 11, 19 and 25. Furthermore, our investigation shows that these inverse LA-semigroups are non-commutative, non-associative and non-left permutable. Table 7 provides presentations of the inverse LA-semigroups that we have been able to find/determine.

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Table 7. Inverse LA-semigroups of specific class

Order	Presentation
5	$< a, b: a^5 = a, aa^3 = b = ba >$
9	$< a, b: a^9 =, aa^7 = b = ba, bb = a^7 >$
11	$< a, b: a^{11} = a, aa^9 = b = ba, bb = a^8 >$
19	$< a, b: a^{19} = a, aa^{17} = b = ba, bb = a^{12} >$
25	$< a, b: a^{25} = a, aa^{23} = b = ba, bb = a^{15} >$

DIRECT AND WREATH PRODUCTS OF INVERSE LA-SEMIGROUPS

If two monoids M_1 and M_2 are described by the presentations $< A_1 \mid R_1 >$ and $< A_2 \mid R_2 >$ respectively, then the direct product $M_1 \times M_2$ has the presentation

$$< A_1, A_2 \mid R_1, R_2, a_1a_2 = a_2a_1, a_1 \in A_1, a_2 \in A_2 >, \quad (1)$$

while the free product $M_1 * M_2$ has the presentation $< A_1, A_2 \mid R_1, R_2 >$.

The structure of the direct product is well known in group theory. If $G^1 = < A_i \mid R_j >$ and $G^2 = < B_j \mid R_j >$ are two groups with no common element except the identity element, then the presentation for the direct product includes all the generators and relations of G_1 and G_2, along with relations $A_i^{-1}B_j^{-1}A_iB_j = e$, $(i = 1, \ldots, m;\ j = 1, \ldots, n)$. We follow the same approach for the inverse LA-semigroups, but the composition makes a substantial difference. The direct product of groups requires two or more groups with identity and inverses. To accommodate this essentiality, we consider two inverse LA-semigroups $€_1$ and $€_2$, both with inverses of the generators, with at least one of them having the left identity e. Also, we take extra relations having specific orders $(a_i^{-1}b_j^{-1})(a_ib_j) = e, (i = 1, \ldots, m;\ j = 1, \ldots, n)$, which follow the left invertive law.

Theorem 2. Let $€_1$ be an inverse LA-semigroup given by the presentation $< A \mid R_1 >$ and $€_2$ be an inverse LA-monoid containing a left identity e given by the presentation $< B \mid R_2 >$. Then the presentation $< A, B \mid R_1, R_2, (a^{-1}b^{-1})(ab) = e\ (a \in A, b \in B) >$ of the direct product of $€_1$ and $€_2$ is an inverse LA-semigroup.

Proof. Since the left invertive law and the left inverse for each element hold in every inverse LA-semigroup, then by including the relations $(a^{-1}b^{-1})(ab) = e$, the direct product of $€_1$ and $€_2$ becomes an inverse LA-semigroup. □

Example 4. Consider an inverse LA-semigroup $€_1 =< a, b \mid aa^2 = b = ba, a^4 = a >$ and an inverse LA-monoid $€_2 =< c \mid c^5 = c, cc^3 = c^4 >$ containing the left identity c^4. Then by using Theorem 2, we have $(a^{-1}c^{-1})(ac) = c^4$and $(b^{-1}c^{-1})(bc) = c^4$, which implies that $(a^3c^3)(ac) = c^4$ and $(a^2c^3)(bc) = c^4$. Using the medial law, we have $(a^3a)(c^3c) = c^4$ and $(a^2b)(c^3c) = c^4$. Thus, the direct product $€_1 \times €_2$ transforms it into an inverse LA-semigroup as given in Table 8. The presentation of the direct product of $€_1$ and $€_2$ is given by $€_1 \times €_2 =< a, b, c \mid aa^2 = b = ba, a^4 = a, c^5 = c, cc^3 = c^4, (a^3a)(c^3c) = c^4, (a^2b)(c^3c) = c^4 >$.

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Table 8. Direct product of $€_1$ and $€_2$

	a	a^2	a^3	b	c	c^2	c^3	c^4
a	a^2	b	a^3	a	c	c^2	c^3	c^4
a^2	a^3	a	a^2	b	c	c^2	c^3	c^4
a^3	a	a^3	b	a^2	c	c^2	c^3	c^4
b	b	a^2	a	a^3	c	c^2	c^3	c^4
c	c^3	c^3	c^3	c^3	c^2	c	c^4	c^3
c^2	c^2	c^2	c^2	c^2	c^3	c^4	c	c^2
c^3	c	c	c	c	c^4	c^3	c^2	c
c^4	c^4	c^4	c^4	c^4	c	c^2	c^3	c^4

We now define a presentation for the wreath product of two inverse LA-semigroups. Let $\boldsymbol{L}$ and $\boldsymbol{U}$ be two LA-monoids. Then $\boldsymbol{L}^{\times \boldsymbol{U}}$ is the set of all functions from $\boldsymbol{U}$ into $\boldsymbol{L}$, while $\boldsymbol{L}^{\oplus \mathbf{U}}$ is the set of all such functions f with finite support, i.e. satisfying the rule that $(x)f = 1_L$ for all $x \in \boldsymbol{U}$.

Let € be an inverse LA-semigroup and $€_M$ be an inverse LA-monoid. The Cartesian product $€^{\times €_M} \times €_M$ is called the unrestricted wreath product of € by $€_M$. We define a binary operation in $€^{\times €_M} \times €_M$ by

$$(f,m)(g,m') = \left(fg^{mm'}, mm'\right), \tag{2}$$

where $g^m: €_M \to €$ is defined by

$$(x)g^m = (xm)g \quad \text{for all } x \in €_M; \tag{3}$$

$fg^{tt'}: €_M \to €$ is defined by

$$(x)\, fg^{tt'} = [I_U].[(tt')I_U.(x)f]g \quad \text{for all } x \in €_M; \tag{4}$$

and $f^{(tt')u''}g^{tt'}: €_M \to €$ is defined by

$$(x)\, f^{(tt')u''}g^{tt'} = [(tt')u''][[(tt')u''.((x)f)]g]. \tag{5}$$

Theorem 3. The unrestricted wreath product of the inverse LA-semigroup € by an inverse LA-monoid $€_M$, i.e. $€^{\times €_M} \times €_M$, is an inverse LA-semigroup with left identity $(\bar{I}, I_{€_M})$.

Proof. The binary operation defined by equation (2) shows that

$$\begin{aligned}[(h,m'')(g,m')](f,m) &= \left(hg^{m''m'}, m''m'\right)(f,m) \\ &= \left(\left(hg^{m''m'}\right)f^{(m''m')m}, (m''m')m\right).\end{aligned} \tag{6}$$

Focusing on the first element of the ordered pair, we get

$$\left(hg^{m''m'}\right)f^{(m''m')m} = \left(f^{(m''m')m}g^{m''m'}\right)h.$$

Using equation (4), we deduce that

$$(x)((f^{(m''m')m}g^{m''m'})h) = [(x)(f^{(m''m')m}g^{m''m'})]h.$$

Furthermore, equation (5) implies that

$$[(x)(f^{(m''m')m}g^{m''m'})]h = \big[[(m''m')m].[[(m''m')m.(x)f]g]\big]h$$

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$$= \Big[[(tt')m''].\,[[(tt')m''.\,(x)f]g]\Big]h$$
$$= \big[[(tt')m''.\,(x)f]g\big]h^{(tt')m''}$$
$$= \big[(x)\big(fg^{(tt')m''}\big)\big]h^{(tt')m''}$$
$$= (x)[\big(fg^{(tt')m''}\big)\})h^{(tt')m''}].$$

Using equation (6), we have

$$[(h,m'')(g,m')](f,m) = ((hg^{m''m'})f^{(m''m')m},(m''m')m)$$
$$= \Big(\big(fg^{(tt')m''}\big)h^{(tt')m''},(tt')m''\Big)$$
$$= [(f,m)(g,m')](h,m''),$$

and

$$(\bar{I},I_{€_M})(f,m) = (\bar{I}f^{I_{€_M}m},I_{€_M}m)$$
$$= (f,m).$$

Hence $€^{\times €_M} \times €_M$ is an inverse LA-semigroup with left identity $(\bar{I},I_{€_M})$. □

The restricted wreath product of the inverse LA-semigroup € by an inverse LA-monoid $€_M$ is the Cartesian product $€^{\oplus €_M} \times €_M$, with the multiplication as defined above.

Corollary 1. The restricted wreath product of the inverse LA-semigroup € by an inverse LA-monoid $€_M$, i.e. $€^{\oplus €_M}$, is an inverse LA-semigroup with left identity $(\bar{I},I_{€_M})$.

CONCLUSIONS

This study has provided us with an opportunity to make use of inverse LA-semigroups in information hiding techniques and graph theory. We have constructed the finite presentations of inverse LA-semigroup, inverse LA-monoid, left permutable inverse LA-semigroup and LA-group. In addition, we have found a generalised presentation of a finite LA-group. We have introduced the direct product of the inverse LA-semigroup by an inverse LA-monoid and proved that the wreath product of an inverse LA-semigroup by an inverse LA-monoid is an inverse LA-monoid.

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