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Full Paper

Generalised (α, β)-derivations in rings with involution

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Abstract: Let R be a ring with involution and let α and β be endomorphisms of R. In this paper we characterise generalised Jordan (α, β)-higher derivations and related maps on (semi)-prime rings with involution. As consequences of our main theorems, many known results can be either generalised or deduced.

Keywords: semiprime ring, involution, generalised (α, β)-derivation, (α, β)-higher derivation, generalised (α, β)-higher derivation

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INTRODUCTION

Throughout this article, unless otherwise mentioned, R will denote an associative ring. A ring endowed with involution $*$ is called a ring with involution, or a $*$-ring. For basic definitions and notations, we refer the reader to Herstein [1, 2]. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. U is also called $*$-closed and square closed if $U^* = U$ and $u^2 \in U$ for all $u \in U$. An additive mapping, $d: R \to R$, is called a derivation (correspondingly, Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (correspondingly, $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. Following Brešar [3], an additive mapping $F: R \to R$ is said to be a generalised derivation (correspondingly, generalised Jordan derivation) on R if there exists a derivation $d: R \to R$ such that $F(xy) = F(x)y + xd(y)$ (correspondingly, $F(x^2) = F(x)x + xd(x)$) holds for all $x, y \in R$.

For given endomorphisms α and β, an additive mapping $d: R \to R$ is said to be an (α, β)-derivation (correspondingly, Jordan (α, β)-derivation) if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ (correspondingly, $d(x^2) = d(x)\alpha(x) + \beta(x)d(x)$) holds for all $x, y \in R$. According to Ashraf et al. [4], an additive mapping $F: R \to R$ is called a generalised (α, β)-derivation (correspondingly, generalised Jordan (α, β)-derivation) on R if there exists an (α, β)-derivation, $d: R \to R$ such that $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ (correspondingly, $F(x^2) = F(x)\alpha(x) + \beta(x)d(x)$) holds for all $x, y \in R$. It is obvious to see that every generalised (α, β)-derivation on a ring is a generalised Jordan (α, β)-derivation, but the converse need not be true in general [5 (Example 3.1)]. A number of authors have studied this problem in the setting of prime and semiprime rings. Recently, Ali and Haetinger [5] proved that every generalised Jordan (α, β)-derivation on a 2-torsion free semiprime ring is a generalised (α, β)-derivation.

The concept of derivations was extended to higher derivations by Hasse and Schmidt [6]. Let $D = \{d_n\}_{n\in \boldsymbol{N}}$ be a family of additive mappings on R. D is said to be a higher derivation (correspondingly, Jordan higher derivation) on R if $d_0 = I_R$(where I_R is the identity map on R) and $d_n(xy) = \sum_{i+j=n} d_i(x)d_j(y)$ (correspondingly, $d_n(x^2) = \sum_{i+j=n} d_i(x)d_j(x)$) for all $x, y \in R$ and for each $n \in N$. It is easy to see that the first member of each higher derivation is itself a derivation. More related results can be found in Haetinger [7]. Later on, Cortes and Haetinger [8] defined generalised higher derivations: a family $F = (f_i)_{i\in \boldsymbol{N}}$ of additive mappings of a ring R, such that $f_0 = I_R$ is said to be a generalised higher derivation (correspondingly, generalised Jordan higher derivation) of R if there exists a higher derivation (correspondingly, Jordan higher derivation) $D = \{d_n\}_{n\in \boldsymbol{N}}$ and for each $n \in N$, $f_n(xy) = \sum_{i+j=n} f_i(x)d_j(y)$ (correspondingly, $f_n(x^2) = \sum_{i+j=n} f_i(x)d_j(x)$) holds for all $x, y \in R$. Obviously, every generalised higher derivation is a generalised Jordan higher derivation, but the converse need not be true. The converse has already been proved by Cortes and Haetinger [8] for square closed Lie ideals of a prime ring R. Later, Wei and Xao [9] established this result for a 2-torsion free semiprime ring. In 2010, Ashraf et al. [10] introduced the concept of (α, β)-higher derivations as follows: a family D of additive mappings d_n on R is said to be an (α, β)-higher derivation (correspondingly, Jordan (α, β)-higher derivation) of R if $d_0 = I_R$ and $d_n(xy) = \sum_{i+j=n} d_i\left(\beta^{n-i}(x)\right) d_j\left(\alpha^{n-j}(y)\right)$(correspondingly, $d_n(x^2) = \sum_{i+j=n} d_i\left(\beta^{n-i}(x)\right) d_j\left(\alpha^{n-j}(x)\right)$) for all $x, y \in R$ and for each $n \in N$. For given endomorphisms α and β, a family $F = (f_i)_{i\in \boldsymbol{N}}$ of additive mappings $f_n: R \to R$ is said to be a generalised (α, β)-higher derivation (correspondingly, generalised Jordan (α, β)-higher derivation) of R if there exists an (α, β)-higher derivation $D = \{d_n\}_{n\in \boldsymbol{N}}$ and for each $n \in N$, $f_n(xy) = \sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right) d_j\left(\alpha^{n-j}(y)\right)$ (correspondingly, $f_n(x^2) = \sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right) d_j\left(\alpha^{n-j}(x)\right)$) holds for all $x, y \in R$. It is straightforward to check that any generalised (α, β)-higher derivation is a generalised Jordan (α, β)-higher derivation. However, the converse statement need not be true. Ashraf and Khan [11] proved that every generalised Jordan (α, β)-higher derivation is a generalised (α, β)-higher derivation on Lie ideals of a prime ring R. We study these problems in the setting of semiprime rings with involution (Theorem 1).

Let R be a $*$-ring. An additive mapping, $d: R \to R$, is said to be a $*$-derivation (correspondingly, Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (correspondingly, $d(x^2) = d(x)x^* + xd(x)$) for all $x, y \in R$. These mappings appear naturally in the theory of representability of quadratic forms by bilinear forms. Following Ali [12], $F: R \to R$ is called a generalised $*$-derivation (correspondingly, generalised Jordan $*$-derivation) if there exists a $*$-derivation (correspondingly, Jordan $*$-derivation) $d: R \to R$ such that $F(xy) = F(x)y^* + xd(y)$ (correspondingly, $F(x^2) = F(x)x^* + xd(x)$) holds for all $x, y \in R$. Several authors characterised the additive mappings satisfying $*$-differential identities in the setting of prime and semiprime rings.

In 1976 Herstein [2] proved the following result: let R be a simple ring with $char(\mathrm{R}) \neq 2$ such that $\dim_Z \mathrm{R} > 4$. Let $\mathrm{d} : \mathrm{R} \to \mathrm{R}$ be such that $\mathrm{d}(\mathrm{xx}^*) = \mathrm{d}(\mathrm{x})\mathrm{x}^* + \mathrm{xd}(\mathrm{x}^*)$ for all $x \in \mathrm{R}$. Then d is a derivation of R. Later Daif and El-Sayiad [13] obtained the following result: let R be a 2-torsion free semiprime $*$-ring and $F: R \to R$ be an additive mapping associated with a derivation $d: R \to R$ such that $F(xx^*) = F(x)x^* + xd(x^*)$ holds for all $x \in R$. Then F is a generalised Jordan derivation. In the same manner, Ashraf et al. [14] established Daif and El-Sayiad's result in a more general form and proved the following: let R be a 2-torsion free semiprime $*$-ring. Suppose that α and β are endomorphisms of R such that α is an automorphism of R. If there exists an additive mapping $F: R \to R$ associated with an (α, β)-derivation d of R such that $F(xx^*) = F(x)\alpha(x^*) + \beta(x)d(x^*)$ holds for all $x \in R$, then $F(xy) = F(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$. Besides proving the above-mentioned result, they proved another interesting result as follows: let R be a 2-torsion free semiprime $*$-ring. Suppose that α and β are endomorphisms of R such that α is an automorphism of R. If there exists an additive mapping $F: R \to R$ associated with an (α, β)-derivation d of R such that $F(xy^*x) = F(x)\alpha(y^*x) + \beta(x)d(y^*)\alpha(x) + \beta(xy^*)d(x)$ holds for all $x, y \in R$, then F is a generalised (α, β)-derivation. Recently, Ezzat [15] studied these results on generalised higher derivations. In this paper apart from proving some other results, we study the above-mentioned theorem in the setting of generalised (α, β)-higher derivations on a semiprime ring with involution.

GENERALISED (α, β)-HIGHER DERIVATIONS ON SEMIPRIME RINGS

We begin our discussion with the following key lemmas which will be extensively useful in proving the main results.

Lemma 1 [14 (Lemma 2.3)]**.** Let R be a 2-torsion free semiprime ring with involution. Suppose that α is an automorphism of R. If there exists an element $a \in R$ such that $a\alpha(x^*) = a\alpha(x)$ holds for all $x \in R$, then $a \in Z(R)$.

It is easy to prove the next lemma by using the same techniques used in Lemma 1

Lemma 2. Let R be a prime ring with involution with $char(R) \neq 2$ and let U be a noncentral $*$-closed Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that α is an automorphism of R. If there exists an element $a \in U$ such that $a\alpha(u^*) = a\alpha(u)$ holds for all $u \in U$, then $a \in Z(R)$.

The first main result of the present paper is the following theorem.

Theorem 1. Let R be a 2-torsion free semirprime ring with involution. Suppose there exists a family of additive mappings $F = \{f_i\}_{i \in N}$ of R associated with some (α, β)-higher derivation $D = \{d_n\}_{n \in N}$ of R, where α and β are commuting automorphisms on R such that $f_n(xx^*) = \sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right) d_j\left(\alpha^{n-j}(x^*)\right)$ holds for all $x \in R$ and for each $n \in N$. Then F is a generalised (α, β)-higher derivation.

Proof. By assumption, we have

$$f_n(xx^*) = \sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right) d_j\left(\alpha^{n-j}(x^*)\right) \tag{1}$$

for all $x \in R$. Linearisation of the above expression yields

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$$f_{\mathrm{n}}(xy^* + yx^*) = \sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right) d_j\left(\alpha^{n-j}(y^*)\right) + \sum_{i+j=n} f_i\left(\beta^{n-i}(y)\right) d_j\left(\alpha^{n-j}(x^*)\right) \quad (2)$$

for all $x, y \in R$. Taking $y = x^*$, we find

$$\eta_n(x) + \eta_n(x^*) = 0 \quad (3)$$

for all $x \in R$, where $\eta_n(x)$ stands for $\sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right) d_j\left(\alpha^{n-j}(x)\right)$. Our aim is to show that $\eta_n(x) = 0$ for all $x \in R$. To show $\eta_n(x) = 0$, we proceed by induction. If $n = 0$, then it is easy to obtain $\eta_0(x) = 0$ for all $x \in R$. If $n = 1$, then the result follows [5, (Theorem 3.1)]. Now suppose $\eta_m(x) = 0$ for $x \in R$ and for all $m < n$. We set $A = f_n(x(xy^* + yx^*) + (xy^* + yx^*)x^*)$. In view of (1), we have

$$\begin{aligned}
A &= \sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right) d_j\left(\alpha^{n-j}(xy^* + yx^*)\right) + f_i\left(\beta^{n-i}(xy^* + yx^*)\right) d_j\left(\alpha^{n-j}(x^*)\right) \\
&= \sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right)\left(\sum_{p+q=j} d_p\left(\beta^{j-p}\alpha^{n-j}(x)\right) d_q\left(\alpha^{j-q}\alpha^{n-j}(y^*)\right)\right. \\
&\quad \left. + \sum_{p+q=j} d_p\left(\beta^{j-p}\alpha^{n-j}(y)\right) d_q\left(\alpha^{j-q}\alpha^{n-j}(x^*)\right)\right) \\
&\quad + \sum_{i+j=n}\left(\sum_{s+t=i} f_s\left(\beta^{i-s}\beta^{n-i}(x)\right) d_t\left(\alpha^{i-t}\beta^{n-i}(y^*)\right)\right. \\
&\quad \left. + \sum_{s+t=i} f_s\left(\beta^{i-s}\beta^{n-i}(y)\right) d_t\left(\alpha^{i-t}\beta^{n-i}(x^*)\right)\right) d_j\left(\alpha^{n-j}(x^*)\right).
\end{aligned}$$

Hence we get

$$\begin{aligned}
A &= \sum_{i+j=n} f_i(\beta^{n-i}(x)) \sum_{p+q=j} d_p\left(\beta^{j-p}\alpha^{n-j}(x)\right) d_q\left(\alpha^{j-q}\,\alpha^{n-j}(y^*)\right) \\
&\quad + \sum_{i+j} f_i(\beta^{n-j}(x)) \sum_{p+q=j} d_p\left(\beta^{j-p}\alpha^{n-j}(y)\right) d_q\left(\alpha^{j-q}\alpha^{n-j}(x^*)\right) \\
&\quad + \sum_{i+j=n}\sum_{s+t=i} f_s\left(\beta^{i-s}\beta^{n-i}(x)\right) d_t\left(\alpha^{i-t}\beta^{n-i}(y^*)\right) d_j\left(\alpha^{n-j}(x^*)\right) \\
&\quad + \sum_{i+j=n}\sum_{s+t=i} f_s\left(\beta^{i-s}\beta^{n-i}(y)\right) d_t\left(\alpha^{i-t}\beta^{n-i}(x^*)\right) d_j\left(\alpha^{n-j}(x^*)\right).
\end{aligned}$$

This can be written as

$$A = \sum_{i+j=n} f_i(\beta^{n-i}(x)) \sum_{p+q=j} d_p\left(\beta^{j-p}\alpha^{n-j}(x)\right) d_q\left(\alpha^{j-q}\,\alpha^{n-j}(y^*)\right)$$

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$$+\sum_{i+j} f_i(\beta^{n-j}(x)) \sum_{p+q=j} d_p\left(\beta^{j-p}\alpha^{n-j}(y)\right) d_q\left(\alpha^{j-q}\alpha^{n-j}(x^*)\right)$$

$$+\sum_{i+j=n}\sum_{s+t=i} f_s(\beta^{n-s}(x)) d_t\left(\alpha^{i-t}\beta^{n-i}(y^*)\right) d_j\left(\alpha^{n-j}(x^*)\right)$$

$$+\sum_{i+j=n}\sum_{s+t=i} f_s(\beta^{n-s}(y)) d_t\left(\alpha^{i-t}\beta^{n-i}(x^*)\right) d_j\left(\alpha^{n-j}(x^*)\right).$$

In particular,

$$\begin{aligned} A = &\sum_{i+j=n} f_i(\beta^{n-i}(x))\, d_p(\alpha^{n-p}(x))\alpha^n(y^*) \\ &+\sum_{\substack{i+p+q=n \\ i+p\neq n}} f_i(\beta^{n-i}(x))\;\; d_p\left(\beta^q\alpha^i(x)\right) d_q(\alpha^{n-q}(y^*)) \\ &+\sum_{i+j+k=n} f_i\left(\beta^{n-j}(x)\right) d_j\left(\beta^k\alpha^i(y+y^*)\right) d_k\left(\alpha^{n-k}(x^*)\right) \\ &+\sum_{i+j=n}\sum_{s+t=i} f_s(\beta^{n-s}(y)) d_t\left(\alpha^{i-t}\beta^{n-i}(x^*)\right) d_j\left(\alpha^{n-j}(x^*)\right). \end{aligned} \tag{4}$$

On the other hand, A can be written as

$$\begin{aligned} A \;=&\; f_n(x(y+y^*)x^*) + f_n(x^2y^* + y(x^*)^2) \\ =&\; f_n(x(y+y^*)x^*) + \sum_{i+j=n} f\_i\left(\beta^{n-i}(x^2)\right) d_j\left(\alpha^{n-j}(y^*)\right) \\ &+ \textstyle\sum_{i+j=n} f_i\left(\beta^{n-i}(y)\right) d_j(\alpha^{n-j}(x^*)^2) \\ =&\; f_n(x(y+y^*)x^*) + \; f_n(x^2)\alpha^n(y^*) \\ &+\sum_{\substack{i+p+q=n \\ i+p\neq n}} \mathrm{f_i}\left(\mathrm{\beta^{n-i}(x)}\right) \mathrm{d_p}\left(\beta^q\alpha^i(\mathrm{x})\right) \mathrm{d_q}(\alpha^{\mathrm{n-q}}(\mathrm{y}^*)) \\ &+\sum_{i+j=n} f_i(\beta^{n-i}(y)) \sum_{k+l=j} d_k\left(\alpha^{j-k}\beta^{n-j}(x^*)\right) d_l(\alpha^{n-l}(x^*)). \end{aligned} \tag{5}$$

Comparison of (4) and (5) yields

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$$f_n(x(y+y^*)x^*) = -\eta_n(x)\alpha^n(y^*) + \sum_{i+j+k=n} f_i\left(\beta^{n-i}(x)\right) d_j\left(\beta^k\alpha^i(y+y^*)\right) d_k\left(\alpha^{n-k}(x^*)\right)$$

for all $x, y \in R$. Taking y as $y - y^*$, we find that

$$\eta_n(x)\alpha^n(y^*) = \eta_n\alpha^n(y). \tag{6}$$

In view of Lemma 1, we find $\eta_n(x) \in Z(R)$ for all $x \in R$. Next, putting y as y^* in (2), we obtain

$$f_n(xy + y^*x^*) = \textstyle\sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right) d_j\left(\alpha^{n-j}(y)\right) \sum_{i+j=n} f_i\left(\beta^{n-i}(y^*)\right) d_j\left(\alpha^{n-j}(x^*)\right) \tag{7}$$

for all $x, y \in R$. Replacing y by xy in the above expression, we get

$$\begin{aligned}
f_n(x^2y + y^*x^{*2}) &= \sum_{i+j=n} f_i(\beta^{n-i}(x))d_j(\alpha^{n-j}(xy)) + \sum_{i+j=n} f_i\left(\beta^{n-i}(y^*x^*)\right) d_j\left(\alpha^{n-j}(x^*)\right) \\
&= \sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right) \sum_{p+q=j} d_p\left(\beta^{j-p}\alpha^{n-j}(x)\right) d_q\left(\alpha^{j-q}\alpha^{n-j}(y)\right) \\
&\quad + \sum_{i+j=n} f_i\left(\beta^{n-i}(y^*x^*)\right) d_j\left(\alpha^{n-j}(x^*)\right) \\
&= \sum_{i+p=n} f_i\left(\beta^{n-i}(x)\right) d_p(\alpha^{n-p}(x))\alpha^n(y) \\
&\quad + \sum_{\substack{i+p+q=n \\ i+p\neq n}} f_i\left(\beta^{n-i}(x)\right) d_p\left(\beta^q\alpha^i(x)\right) d_q(\alpha^{n-q}(y)) \\
&\quad + f_n(y^*x^*)\alpha^n(x^*) + \sum_{\substack{i+j=n \\ i\neq n}} f_i\left(\beta^{n-i}(y^*x^*)\right) d_j\left(\alpha^{n-j}(x^*)\right)
\end{aligned}$$

for all $x, y \in R$. On the other hand, replacement of x by x^2 in (7) gives

$$\begin{aligned}
f_n(x^2y + y^*x^{*2}) &= \sum_{i+j=n} f_i\left(\beta^{n-i}(x^2)\right) d_j\left(\alpha^{n-j}(y)\right) + \sum_{i+j=n} f_i\left(\beta^{n-i}(y^*)\right) d_j\left(\alpha^{n-j}(x^{*2})\right) \\
&= f_n(x^2)\alpha^n(y) + \sum_{\substack{i+p+q \\ i\neq n}} f_i\left(\beta^{n-i}(x)\right) d_p\left(\beta^q\alpha^i(x)\right) d_q(\alpha^{n-q}(y)) \\
&\quad + \sum_{\substack{s+t+j=n \\ s+t\neq n}} f_s(\beta^{n-s}(y^*))d_t\left(\alpha^j\beta^s(x^*)\right) d_j\left(\alpha^{n-j}(x^*)\right)
\end{aligned}$$

$$+\sum_{s+t=n} f_s\big(\beta^{n-s}(y^*)\big)d_t\big(\alpha^{n-t}(x^*)\big)\alpha^n(x^*)$$

for all $x, y \in R$. In view of the last two relations, we have

$$0 = \eta_n(x)\alpha^n(y) + \left((-f_n(y^*x^*) + \sum_{s+t=n} f_s\big(\beta^{n-s}(y^*)\big)d_t\big(\alpha^{n-t}(x^*)\big)\right)\alpha^n(x^*)$$
$$-\sum_{\substack{s+t+j=n\\ s+t\neq n}} f_s\big(\beta^{n-s}(y^*)\big)d_t\left(\alpha^j\beta^s(x^*)\right)d_j\left(\alpha^{n-j}(x^*)\right)$$
$$-\sum_{\substack{i+j=n\\ i\neq n}} f_i\left(\beta^{n-i}(y^*x^*)\right)d_j\left(\alpha^{n-j}(x^*)\right)$$

for all $x, y \in R$. Putting $x = y$ yields

$$\eta_n(x)\alpha^n(x) - \eta_n(x^*)\alpha^n(x^*) = 0 \tag{8}$$

for all $x \in R$. From (3), we have

$$\eta_n(x)\alpha^n(x) + \eta_n(x^*)\alpha^n(x^*) = 0 \tag{9}$$

for all $x \in R$. Combining the last two relations gives

$$\eta_n(x)\alpha^n(x) = 0 \tag{10}$$

for all $x \in R$. Linearisation of (10) yields

$$\eta_n(x)\alpha^n(y) + \mu(x,y)\alpha^n(x) + \eta_n(y)\alpha^n(x) + \mu(x,y)\alpha^n(y) = 0 \tag{11}$$

for all $x, y \in R$, where $\mu(x, y) = f_n(xy + yx) - \sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right)d_j\left(\alpha^{n-j}(y)\right) + \sum_{i+j=n} f_i\left(\beta^{n-i}(y)\right)d_j\left(\alpha^{n-j}(x)\right)$ for all $x, y \in R$. Putting $x = -x$ in (11) and combining the obtained relation, we get

$$\eta_n(x)\alpha^n(y) + \mu(x,y)\alpha^n(x) = 0$$

for all $x, y \in R$. On right multiplying by $\eta(x)$ and using (10) and Lemma 1, we get

$$\eta_n(x)\alpha^n(y)\eta_n(x) = 0$$

for all $x, y \in R$. Since α^n is an automorphism and R is semiprime, $\eta(x) = 0$ for all $x \in R$, i.e. $f_n(x^2) = \sum_{i+j=n} f_i\left(\beta^{n-i}(x)\right)d_i\left(\alpha^{n-j}(x)\right)$ for all $x \in R$. Hence F is a generalised (α,β)-higher derivation [11, Theorem 2.1 for U = R]. This completely proves the theorem.

As special cases of the above theorem, which are of independent interest, the following corollaries can be made.

Corollary 1 [15, Theorem 2.3]**.** Let R be a 2-torsion free semiprime $*$-ring. Suppose there exists a family of additive mappings $F = \{f_i\}_{i\in N}$ of R associated with some higher derivation $D = \{d_i\}_{i\in N}$ of R such that $f_0 = id_R$, and the relation $f_n(xx^*) = \sum_{i+j=n} f_i(x)d_i(x^*)$ holds for all $x \in R$ and for each $n \in N$. Then F is a generalised higher derivation.

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Corollary 2. Let R be a 2-torsion free semiprime $*$-ring. Suppose there exists a family of additive mappings $D = \{d_i\}_{i\in N}$ of R associated with some higher derivation $D = \{d_i\}_{i\in N}$ of R, where α and β are commuting automorphisms on R such that $d_n(xx^*) = \sum_{i+j=n} d_i\left(\beta^{n-i}(x)\right) d_i(\alpha^{n-j}(x^*))$ holds for all $x \in R$ and for each $n \in N$. Then d is an (α,β)-higher derivation.

GENERALISED (α,β)-DERIVATIONS ON LIE IDEALS OF PRIME RINGS

In this section we deal with the study of generalised (α,β)-derivations on Lie ideals of prime rings, motivated by the work of Ashraf et al. [4, 14]. In particular, we extend some of the proved results [14] to square closed Lie ideals of a prime ring R. We begin with the following theorem.

Theorem 2. Let R be a prime ring with involution with $char(R) \neq 2$ and let U be a non-central $*$-closed Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that α and β are endomorphisms of R such that α is an automorphism of R. If there exists an additive mapping $F: R \to R$ associated with a non-zero (α,β)-derivation d of U such that $F(uu^*) = F(u)\alpha(u^*) + \beta(u)d(u^*)$ holds for all $u \in U$, then F is a generalised (α,β)-derivation.

Proof: We have

$$F(uu^*) = F(u)\alpha(u^*) + \beta(u)d(u^*)$$

for all $u \in U$. Linearisation of the above relation yields

$$F(uv^* + vu^*) = F(u)\alpha(v^*) + F(v)\alpha(u^*) + \beta(v)d(u^*) + \beta(u)d(v^*) \tag{12}$$

for all $u, v \in U$. Replacing u^* for v in (12), we get

$$F(uu + u^*u^*) = F(u)\alpha(u) + F(u^*)\alpha(u^*) + \beta(u^*)d(u^*) + \beta(u)d(u) \tag{13}$$

for all $u \in U$. Furthermore, this can be written as

$$\eta(u) + \eta(u^*) = 0 \tag{14}$$

for all $u \in U$, where $\eta(u) = F(u^2) - F(u)\alpha(u) - \beta(u)d(u)$. Now substituting $2(uv^* + vu^*)$ for v in (12), we get

$$\begin{aligned} 2F(u(uv^* + vu^*) + (uv^* + vu^*)u^*) = {} & 2(F(u)\alpha(uv^* + vu^*) + F(uv^* + vu^*)\alpha(u^*) \\ & + \beta(uv^* + vu^*)d(u^*) + \beta(u)d(uv^* + vu^*)) \end{aligned}$$

for all $u, v \in U$. Since $char(R) \neq 2$, we find that

$$\begin{aligned} F(u(uv^* + vu^*) + (uv^* + vu^*)u^*) = {} & F(u)\alpha(uv^* + vu^*) + F(uv^* + vu^*)\alpha(u^*) \\ & + \beta(uv^* + vu^*)d(u^*) + \beta(u)d(uv^* + vu^*) \end{aligned} \tag{15}$$

for all $u, v \in U$. This implies that

$$\begin{aligned} F(u(uv^* + vu^*) + (uv^* + vu^*)u^*) = {} & F(u)\alpha(uv^*) + F(u)\alpha(vu^*) + F(u)\alpha(v^*)\alpha(u^*) \\ & + F(v)\alpha(u^{*2}) + \beta(u)d(v^*)\alpha(u^*) \\ & + \beta(v)d(u^*)\alpha(u^*) + \beta(uv^*)d(u^*) \\ & + \beta(vu^*)d(u^*) + \beta(u)d(u)\alpha(v^*) \\ & + \beta(u^2)d(v^*) + \beta(u)d(v)\alpha(u^*) + \beta(uv)d(u^*) \end{aligned}$$

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for all $u, v \in U$. On the other hand, we have

$$F(u(uv^* + vu^*) + (uv^* + vu^*)u^*) = F\big(u^2v^* + vu^{*2}\big) + F(uvu^* + uv^*u^*) \tag{16}$$
$$= F(u^2)\alpha(v^*) + F(v)\alpha(u^{*2})$$
$$+ \beta(u^2)d(v^*) + \beta(v)d(u^*)\alpha(u^*)$$
$$+ \beta(vu^*)d(u^*) + F(u(v + v^*)u^*)$$

for all $u, v \in U$. Combining (15) and (16), we obtain

$$F(u(v + v^*)u^*) = -\eta(u)\alpha(v^*) + F(u)\alpha\big((v + v^*)u^*\big) + \beta(u)d\big((v + v^*)u^*\big) \tag{17}$$

for all $u, v \in U$. Replacing v with $v - v^*$ in (17), we get

$$\eta(u)\alpha(v) = \eta(u)\alpha(v^*) \tag{18}$$

for all $u, v \in U$. Lemma 2 implies that $\eta(u) \in Z(R)$ for all $u \in U$. Furthermore, replacing v by v^* in (12), we get

$$F(uv + v^*u^*) = F(u)\alpha(v) + F(v^*)\alpha(u^*) + \beta(v^*)d(u^*) + \beta(u)d(v) \tag{19}$$

for all $u, v \in U$. Now substituting $2uv$ for v in (19) and using the fact that $char(R) \neq 2$, we obtain

$$F\big(u^2v + v^*u^{*2}\big) = F(u)\alpha(uv) + F(v^*u^*)\alpha(u^*) + \beta(v^*u^*)d(u^*) + \beta(u)d(uv) \tag{20}$$

for all $u, v \in U$. On the other hand, taking u^2 for u in (19), we find that

$$F(u^2v + v^*u^{*2}) = F(u^2)\alpha(v) + F(v^*)\alpha(u^{*2}) + \beta(v^*)d(u^{*2}) + \beta(u^2)d(v) \tag{21}$$

for all $u, v \in U$. In view of (20) and (21), we obtain

$$\eta(u)\alpha(v) + \big(F(v^*)\alpha(u^*) - F(v^*u^*) + \beta(v^*)d(u^*)\big)\alpha(u^*) = 0 \tag{22}$$

for all $u, v \in U$. Taking $u = v$ in (22), we get

$$\eta(u)\alpha(u) - \eta(u^*)\alpha(u^*) = 0 \tag{23}$$

for all $u \in U$. In view of (14), we have

$$\eta(u)\alpha(u + u^*) = 0 \tag{24}$$

for all $u \in U$. Substituting u for v in (18), we obtain

$$\eta(u)\alpha(u - u^*) = 0 \tag{25}$$

for all $u \in U$. Comparing (24) and (25) and using the fact that $char(R) \neq 2$, we get

$$\eta(u)\alpha(u) = 0 \tag{26}$$

for all $u \in U$. Since $\eta(u) \in Z(R)$ for all $u \in U$, the last expression implies that $\alpha(u)\eta(u) = 0$ for all $u \in U$. Linearisation of (26) yields

$$\eta(u)\alpha(v) + \eta(v)\alpha(u) + B(u, v)\alpha(u) + B(u, v)\alpha(v) = 0 \tag{27}$$

for all $u, v \in$ U, where $B(u, v) = F(uv + vu) - F(u)\alpha(v) - F(v)\alpha(u) - \beta(u)d(v) - \beta(v)d(u)$. Taking $u = -u$ in (27), we get

$$\eta(u)\alpha(v) - \eta(v)\alpha(u) + B(u, v)\alpha(u) - B(u, v)\alpha(v) = 0 \tag{28}$$

for all $u, v \in U$. Combining (27) and (28) and using the fact that $char(R) \neq 2$, we obtain $\eta(u)\alpha(v) + B(u,v)\alpha(u) = 0$ for all $u, v \in U$, i.e. $\eta(u) = F(u^2) - F(u)\alpha(u) - \beta(u)d(u)$ for all $u \in U$. On right multiplying by $\eta(u)$to the last relation and using the fact $\alpha(u)\eta(u) = 0$ for all $u \in U$, we find that $\eta(u)\alpha(v)\eta(u) = 0$ for all $u, v \in U$. Since α is onto, so we write $\alpha(v) = v$ and from the last relation we conclude that $\eta(u)U\,\eta(u) = (0)$ for all $u \in U$. In view of Corollary 2.1 [16], we conclude that $\eta(u) = 0$ for all $u \in U$, i.e. $\eta(u) = F(u^2) - F(u)\alpha(u) - \beta(u)d(u)$ for all $u \in U$. Therefore, F is a generalised Jordan (α,β)-derivation on U. Hence F is a generalised (α,β)-derivation [4 (Theorem 2.1)]. This completes the proof of the theorem.

As immediate consequences of Theorem 2 we have the following results.

Corollary 3 [5 (Theorem 3.1)]**.** Let R be a prime ring with involution with $char(R) \neq 2$. Suppose that α and β are endomorphisms of R such that α is an automorphism of R. If there exists an additive mapping $F: R \to R$ associated with an (α,β)-derivation d of R such that $F(xx^*) = F(x)\alpha(x^*) + \beta(x)d(x^*)$ for all $x \in R$, then F is a generalised (α,β)-derivation.

Corollary 4 [2, (Theorem 4.1.2)]**.** Let R be a simple ring with $char(R) \neq 2$ such that $\dim_Z R > 4$. Let $d: R \to R$ be such that $d(xx^*) = d(x)x^* + xd(x^*)$ holds for all $x \in R$. Then d is a derivation.

We now prove another theorem in the spirit of Theorem 2.

Theorem 3. Let R be a prime ring with involution with $char(R) \neq 2$ and let U be a non-central $*$-closed Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Suppose that α, β are endomorphisms of R such that α is an automorphism of R. If there exists an additive mapping $F: R \to R$ associated with an (α,β)-derivation d of R such that $F(uv^*u) = F(u)\alpha(v^*u) + \beta(u)d(v^*)\alpha(u) + \beta(uv^*)d(u)$ for all $u, v \in U$, then F is a generalised (α,β)-derivation.

Proof: We have

$$F(uv^*u) = F(u)\alpha(v^*u) + \beta(u)d(v^*)\alpha(u) + \beta(uv^*)d(u) \tag{29}$$

for all $u, v \in U$. Linearising the above expression, we get

$$\begin{aligned} F((u+w)v^*(u+w)) = {} & F(u)\alpha(v^*u) + F(u)\alpha(v^*w) \\ & + F(w)\alpha(v^*u) + F(w)\alpha(v^*w) \\ & + \beta(u)d(v^*)\alpha(u) + \beta(u)d(v^*)\alpha(w) \\ & + \beta(w)d(v^*)\alpha(u) + \beta(w)d(v^*)\alpha(w) \\ & + \beta(uv^*)d(u) + \beta(uv^*)d(w) \\ & + \beta(wv^*)d(u) + \beta(wv^*)d(w) \end{aligned} \tag{30}$$

for all $u, v, w \in U$. On the other hand, we obtain

$$\begin{aligned} F((u+w)v^*(u+w)) = {} & F(uv\hat{}w + wv^*u) + F(u)\alpha(v^*u) \\ & + \beta(u)d(v^*)\alpha(u) + \beta(uv^*)d(u) \\ & + F(w)\alpha(v^*w) + \beta(w)d(v^*)\alpha(w) \\ & + \beta(wv^*)d(w) \end{aligned} \tag{31}$$

for all $u, v, w \in U$. Comparing (30) and (31), we get

$$\begin{aligned} F(uv^*w + wv^*u) = {} & F(u)\alpha(v^*w) + F(w)\alpha(v^*u) + \beta(u)d(v^*)\alpha(w) \\ & + \beta(w)d(v^*)\alpha(u) + \beta(uv^*)d(w) + \beta(wv^*)d(u) \end{aligned} \tag{32}$$

for all $u, v \in U$. Substituting u^2 for w in (32), we obtain

$$F(uv^*u^2 + u^2\, v^*u) = F(u)\alpha(v^*u^2) + F(u^2)\alpha(v^*u) + \beta(u)d(v^*)\alpha(u^2) + \beta(u^2)d(v^*)\alpha(u) + \beta(uv^*)d(u^2) + \beta(u^2v^*)d(u) \qquad (33)$$

for all $u, v \in U$. Further, on replacing $2(uv^* + v^*u)$ for v in (29), we obtain

$$2F(uv^*u^2 + u^2v^*u) = 2(F(u)\alpha(uv^*u) + F(u)\alpha(v^*u^2) + \beta(u)d(uv^*)\alpha(u) + \beta(u)d(v^*u)\alpha(u) + \beta(u^2v^*)d(u) + \beta(uv^*u)d(u))$$

for all $u, v \in U$. Since $char(R) \neq 2$, we find that

$$F(uv^*u^2 + u^2v^*u) = F(u)\alpha(uv^*u) + F(u)\alpha(v^*u^2) + \beta(u)d(uv^*)\alpha(u) + \beta(u)d(v^*u)\alpha(u) + \beta(u^2v^*)d(u) + \beta(uv^*u)d(u) \qquad (34)$$

for all $u, v \in U$. On comparing (33) and (34), we obtain

$$F(u^2)\alpha(v^*u) - F(u)\alpha(u)\alpha(v^*u) - \beta(u)d(u)\alpha(v^*u) = 0 \qquad (35)$$

for all $u, v \in U$. If we take $A(u) = F(u^2) - F(u)\alpha(u) - \beta(u)d(u)$, then the relation (35) reduces to

$$A(u)\alpha(v^*u) = 0 \qquad (36)$$

for all $u, v \in U$. Since α is surjective, (36) implies that

$$\alpha^{-1}\big(A(u)\big)v^*u = 0 \qquad (37)$$

for all $u, v \in U$. Substituting $2v^*u^*$ for v in (37) and using the fact that $char(R) \neq 2$, we find that

$$\alpha^{-1}(A(u))uvu = 0 \qquad (38)$$

for all $u, v \in U$. Since $char(R) \neq 2$ and U is *-closed and square closed Lie ideal of R, the above relation yields

$$\alpha^{-1}\big(A(u)\big)u\alpha^{-1}(w)\alpha^{-1}(A(u))u = 0 \qquad (39)$$

for all $u, w \in U$. This implies that

$$A(u)\alpha(u)wA(u)\alpha(u) = 0 \qquad (40)$$

for all $u, w \in U$. That is

$$A(u)\alpha(u)UA(u)\alpha(u) = (0) \qquad (41)$$

for all $u \in U$. By Corollary 2.1 [16], we get

$$A(u)\alpha(u) = 0 \qquad (42)$$

for all $u \in U$. Linearisation of (42) gives

$$A(u)\alpha(v) + \gamma(u,v)\alpha(u) + A(v)\alpha(u) + \gamma(u,v)\alpha(v) = 0 \qquad (43)$$

for all $u, v \in U$, where $\gamma(u,v) = F(uv + vu) - F(u)\alpha(v) - \beta(u)d(v) - F(v)\alpha(u) - \beta(v)d(u)$ for all $u, v \in U$. Replacing u with $-u$ in (43), we obtain

$$A(u)\alpha(v) + \gamma(u,v)\alpha(u) - A(v)\alpha(u) - \gamma(u,v)\alpha(v) = 0 \qquad (44)$$

for all $u, v \in U$. Combining (43) and (44) and using the fact that $char(R) \neq 2$, we get

$$A(u)\alpha(v) + \gamma(u,v)\alpha(u) = 0 \qquad (45)$$

for all $u, v \in U$. On right multiplication of the above equation by $A(u)$, we obtain

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$$A(u)\alpha(v)A(u) + \gamma(u,v)\alpha(u)A(u) = 0 \tag{46}$$

for all $u, v \in U$. Substituting v^* for v in (46), we find that $\alpha(u)A(u)\alpha(v)\alpha(u)A(u) = 0$. That is

$$\alpha(u)A(u)U\alpha(u)A(u) = (0) \tag{47}$$

for all $u \in U$. Again, by Corollary 2.1 [16], we conclude that

$$\alpha(u)A(u) = 0 \tag{48}$$

for all $u \in U$. Using (48) in (44), we have $A(u)\alpha(v)A(u) = 0$, i.e. $A(u)UA(u) = (0)$ for all $u \in U$. By Corollary 2.1 [16], we obtain $A(u) = 0$ for all $u \in U$, i.e. $F(u^2) - F(u)\alpha(u) - \beta(u)d(u) = 0$ for all $u \in U$. Therefore, F is a generalised Jordan (α, β)-derivation on U. Hence F is a generalised (α, β)-derivation [4 (Theorem 2.1)]. This completes the proof.

Direct application of Theorem 3 yields the following results.

Corollary 5. Let R be a prime ring with involution with $char(R) \neq 2$. Suppose that α and β are endomorphisms of R such that α is an automorphism of R. If there exists an additive mapping $F: R \to R$ associated with an (α, β)-derivation d of R such that $F(xy^*x) = F(x)\alpha(y^*x) + \beta(x)d(y^*)\alpha(x) + \beta(xy^*)d(x)$ for all $x, y \in R$, then F is a generalised (α, β)-derivation .

Corollary 6. Let R be a prime ring with involution such that $char(R) \neq 2$, and let U be a non-central $*$-closed Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If there exists an additive mapping $F: R \to R$ associated with a derivation d of R such that $F(uv^*u) = F(u)v^*u + ud(v^*)u + uv^*d(u)$ for all $u, v \in U$, then F is a generalised derivation.

CONCLUSIONS

In this paper we have studied some functional identities involving certain types of derivations, viz. generalised (α, β)-higher derivations and generalised (α, β)-derivations on semiprime rings with involution. Purely algebraic methods have been used to describe the forms of additive maps applying to rings and their appropriate subsets (Lie ideals). The proposed methods and results are extendable to other classes of algebra, e.g. Banach algebra, Operator algebra and C*-algebra.

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