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Full Paper

Certain new integral inequalities considering the generalised logarithmically (α, m)-preinvexity

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Abstract: Based on an integral identity, we establish certain new k-fractional inequalities via generalised logarithmically (α, m)-preinvexity. We also prove several product-type inequalities for this class of mappings with other convex mappings. Finally, applications to some inequalities in connection with special means are given.

Keywords: Hadamard's inequality, Simpson's inequality, generalised logarithmically (α, m)-preinvex functions

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INTRODUCTION

If $g: \mathrm{I} \subseteq \mathrm{R} \to \mathrm{R}$ is a convex mapping and $e_1, e_2 \in \mathrm{I}$ with $e_1 < e_2$, then one has

$$g\left(\frac{e_1+e_2}{2}\right) \le \frac{1}{e_2-e_1}\int_{e_1}^{e_2} g(t)\,\mathrm{d}t \le \frac{g(e_1)+g(e_2)}{2}, \tag{1}$$

which is called the Hadamard's inequality.

The following inequality is named the Simpson's integral inequality:

$$\left|\frac{1}{6}\left[g(e_1)+4g\left(\frac{e_1+e_2}{2}\right)+g(e_2)\right]-\frac{1}{e_2-e_1}\int_{e_1}^{e_2} g(t)\,\mathrm{d}t\right| \le \frac{1}{2880}\left\|g^{(4)}\right\|_\infty (e_2-e_1)^4, \tag{2}$$

where $g:[e_1, e_2] \to \mathrm{R}$ is a four-time continuously differentiable mapping on (e_1, e_2) and $\left\|g^{(4)}\right\|_\infty = \sup_{t\in(e_1,e_2)}\left|g^{(4)}(t)\right| < \infty$.

For recent results about inequalities (1) and (2), we refer to some studies by Khan et al. [1], Sarikaya et al. [2], Awan et al. [3], Dragomir et al. [4], Du et al. [5], Wu et al. [6] and Set et al. [7].

Let us consider an m-invex set A. A set $\mathrm{A} \subseteq \mathrm{R}^n$ is named an m-invex set with respect to the mapping $\eta: \mathrm{A}\times\mathrm{A}\times(0,1] \to \mathrm{R}^n$ if $mx + \lambda\eta(y,x,m) \in \mathrm{A}$ holds for all $x, y \in \mathrm{A}$ and $\lambda \in [0,1]$

with some fixed $m \in (0,1]$. A mapping $g : \mathrm{A} \to \mathrm{R}$ is called generalised quasi-m-preinvex with respect to η if the following inequality

$$g\big(mx + \lambda \eta(y,x,m)\big) \le \max\{g(x), g(y)\}$$

holds for all $\lambda \in [0,1]$ and $x, y \in \mathrm{A}$ with some fixed $m \in (0,1]$.

A mapping $g : \mathrm{A} \to \mathrm{R}$ is called generalised (m,s)-Breckner-preinvex with respect to η if the following inequality

$$g\big(mx + t\eta(y,x,m)\big) \le t^s g(y) + m(1-t)^s g(x)$$

holds for all $t \in [0,1]$ and $x, y \in \mathrm{A}$ with some fixed $s, m \in (0,1]$.

Very recently, an article by Zhang et al. [8] derived the following integral identity via the second-order derivative of g to establish k-fractional integral inequalities for the generalised (m,h)-preinvex functions.

Lemma 1 [8]. Let $\mathrm{A} \subseteq \mathrm{R}$ be an open m-invex subset with respect to $\eta : \mathrm{A} \times \mathrm{A} \times (0,1] \to \mathrm{R} \setminus \{0\}$ for some fixed $m \in (0,1]$ and let $e_1, e_2 \in \mathrm{A}$, $e_1 < e_2$ with $\eta(e_2, e_1, m) > 0$. If $g : \mathrm{A} \to \mathrm{R}$ is twice differentiable on A such that g'' is integrable on $[me_1, me_1 + \eta(e_2, e_1, m)]$, then the following equation for k-fractional integrals with $x \in [e_1, e_2]$, $\lambda \in [0,1]$, $\tau > 0$ and $k > 0$ holds:

$$\mathrm{L}_{g,\eta}(\tau,k;x,\lambda,m,e_1,e_2) = \frac{\eta^{\frac{\tau}{k}+2}(x,e_1,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)} \int_0^1 t\left[\left(\frac{\tau}{k}+1\right)\lambda - t^{\frac{\tau}{k}}\right] g''\big(me_1 + t\eta(x,e_1,m)\big)\,\mathrm{d}t$$

$$+\frac{(-1)^{\frac{\tau}{k}+2}\eta^{\frac{\tau}{k}+2}(x,e_2,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)} \int_0^1 t\left[\left(\frac{\tau}{k}+1\right)\lambda - t^{\frac{\tau}{k}}\right] g''\big(me_2 + t\eta(x,e_2,m)\big)\,\mathrm{d}t,$$

where

$$\begin{aligned}
&\mathrm{L}_{g,\eta}(\tau,k;x,\lambda,m,e_1,e_2)\\
&:= \frac{1-\lambda}{\eta(e_2,e_1,m)}\left[\eta^{\frac{\tau}{k}}(x,e_1,m)g\big(me_1+\eta(x,e_1,m)\big) + (-1)^{\frac{\tau}{k}}\eta^{\frac{\tau}{k}}(x,e_2,m)g\big(me_2+\eta(x,e_2,m)\big)\right]\\
&\quad+\frac{\lambda}{\eta(e_2,e_1,m)}\left[\eta^{\frac{\tau}{k}}(x,e_1,m)g(me_1) + (-1)^{\frac{\tau}{k}}\eta^{\frac{\tau}{k}}(x,e_2,m)g(me_2)\right]\\
&\quad+\frac{\frac{1}{\frac{\tau}{k}+1}-\lambda}{\eta(e_2,e_1,m)}\left[(-1)^{\frac{\tau}{k}+1}\eta^{\frac{\tau}{k}+1}(x,e_2,m)g'\big(me_2+\eta(x,e_2,m)\big) - \eta^{\frac{\tau}{k}+1}(x,e_1,m)g'\big(me_1+\eta(x,e_1,m)\big)\right]\\
&\quad-\frac{\Gamma_k(\tau+k)}{\eta(e_2,e_1,m)}\left[{}_k\mathrm{J}^{\tau}_{(me_1+\eta(x,e_1,m))^-}g(me_1) + {}_k\mathrm{J}^{\tau}_{(me_2+\eta(x,e_2,m))^+}g(me_2)\right]
\end{aligned}$$

and Γ_k is the k-Gamma function.

Let us mention a formal definition for generalised logarithmically (α,m)-preinvex function.

Definition 1 [9]. Let $\mathrm{A} \subseteq \mathrm{R}^n$ be an m-invex set with respect to $\eta : \mathrm{A} \times \mathrm{A} \times (0,1] \to \mathrm{R}^n$. A mapping $g : \mathrm{A} \to \mathrm{R}^+$ is generalised logarithmically (α,m)-preinvex with respect to η if the inequality

$$g\big(mx + \lambda\eta(y,x,m)\big) \le [g(y)]^{\lambda^\alpha}[g(x)]^{m(1-\lambda^\alpha)}$$

holds for all $\lambda \in [0,1]$ and $x, y \in \mathrm{A}$ with some fixed $\alpha, m \in (0,1]$.

For some related results concerning logarithmically functions, we refer to studies by Karabayir et al. [10], Zafar et al. [11], Hussain and Rafeeq [12], Noor et al. [13] and Latif et al. [14].

A recent article by Wu et al. [15] presented some product-type integral inequalities for strongly logarithmically convex mappings, one of them given in the following theorem.

Theorem 1 [15]. Let $e_1, e_2 \in \mathrm{R}_0$ with $e_1 < e_2$, and let $f, g : [e_1, e_2] \to \mathrm{R}^+$. If f is s-convex on $[e_1, e_2]$ with $s \in (0,1]$ and g^q is strongly logarithmically convex on $[e_1, e_2]$ with modulus $c \geq 0$ for $q \geq 1$, then we have

$$\begin{aligned}\frac{1}{e_2 - e_1}\int_{e_1}^{e_2} f(x)g(x)\mathrm{d}x \leq \left[\frac{f(e_1)+f(e_2)}{s+1}\right]^{1-\frac{1}{q}} &\left\{\left[\frac{g^q(e_1)-g^q(e_2)}{s+2}+\frac{g^q(e_2)}{s+1}\right]f(e_1)\right.\\ &\left.+\left[\frac{g^q(e_1)-g^q(e_2)}{(s+1)(s+2)}+\frac{g^q(e_2)}{s+1}\right]f(e_2)-\frac{c(e_2-e_1)^2}{(s+2)(s+3)}[f(e_1)+f(e_2)]\right\}^{\frac{1}{q}}.\end{aligned} \tag{3}$$

Earlier, an article by Kirmaci et al. [16] presented the following inequality of Hadamand type for the product of s-convex mappings.

Theorem 2 [16]. Let $f, g : [e_1, e_2] \to \mathrm{R}$, $e_1, e_2 \in [0, \infty)$, $e_1 < e_2$, be functions such that f, g and fg are in $L^1([e_1, e_2])$. If f is s_1-convex and g is s_2-convex on $[e_1, e_2]$ for some fixed $s_1, s_2 \in (0,1]$, then

$$\begin{aligned}\frac{1}{e_2 - e_1}\int_{e_1}^{e_2} f(x)g(x)\mathrm{d}x &\leq \frac{1}{s_1+s_2+1}M(e_1,e_2) + B(s_1+1, s_2+1)N(e_1,e_2)\\ &= \frac{1}{s_1+s_2+1}\left[M(e_1,e_2)+s_1 s_2\frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2+1)}N(e_1,e_2)\right],\end{aligned} \tag{4}$$

where $M(e_1, e_2) = f(e_1)g(e_1) + f(e_2)g(e_2)$ and $N(e_1, e_2) = f(e_1)g(e_2) + f(e_2)g(e_1)$.

Our aim is to establish, using Lemma 1, some new integral inequalities like those given in the article by Zhang et al. [8], but now for the class of generalised logarithmically (α, m)-preinvex mappings. We also prove analogues of inequalities (3) and (4) for this class of mappings with other convex mappings. Finally, applications of our results to special means are provided.

We end this section by reciting the concept of the k-fractional integral operators:

Definition 2 [17]. Let $h \in L^1([\mu, \nu])$; the k-fractional integrals ${}_k\mathrm{J}_{\mu^+}^{\tau}h(x)$ and ${}_k\mathrm{J}_{\nu^-}^{\tau}h(x)$ of order $\tau > 0$ are defined by

$${}_k\mathrm{J}_{\mu^+}^{\tau}h(x) = \frac{1}{k\Gamma_k(\tau)}\int_{\mu}^{x}(x-\lambda)^{\frac{\tau}{k}-1}h(\lambda)\mathrm{d}\lambda, \quad (0 \leq \mu < x < \nu),$$

$${}_k\mathrm{J}_{\nu^-}^{\tau}h(x) = \frac{1}{k\Gamma_k(\tau)}\int_{x}^{\nu}(\lambda-x)^{\frac{\tau}{k}-1}h(\lambda)\mathrm{d}\lambda, \quad (0 \leq \mu < x < \nu),$$

respectively, where $k > 0$ and Γ_k is the k-gamma function defined as $\Gamma_k(x) = \int_0^{\infty}\lambda^{x-1}e^{-\frac{\lambda^k}{k}}\mathrm{d}\lambda$.

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INEQUALITIES FOR DIFFERENTIABLE FUNCTIONS

Throughout this section, let $A \subseteq R$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \to R \setminus \{0\}$ and let $e_1, e_2 \in A$, $e_1 < e_2$ with $\eta(e_2, e_1, m) > 0$.

We also assume that $g : A \to R^+$ is twice differentiable on A such that g'' is integrable on $[me_1, me_1 + \eta(e_2, e_1, m)]$. Using Lemma 1, we now state the following theorem.

Theorem 3. If $(g'')^q$ is generalised logarithmically (α, m)-preinvex on A for some fixed $\alpha, m \in (0,1]$ and $q \geq 1$, then the following inequality for k-fractional integrals with $x \in [e_1, e_2]$, $\lambda \in [0,1]$, $\tau > 0$, $k > 0$ holds:

$$\left| L_{g,\eta}(\tau, k; x, \lambda, m, e_1, e_2) \right|$$

$$\leq \Delta_1^{1-\frac{1}{q}}(k,\tau,\lambda) \left\{ \left| \frac{\eta^{\frac{\tau}{k}+2}(x, e_1, m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2, e_1, m)} \right| \left([g''(x)]^q \Delta_2(k,\tau,\lambda;\alpha) + [g''(e_1)]^{qm} \Delta_3(k,\tau,\lambda;\alpha) \right)^{\frac{1}{q}} \right.$$

$$\left. + \left| \frac{(-1)^{\frac{\tau}{k}+2} \eta^{\frac{\tau}{k}+2}(x, e_2, m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2, e_1, m)} \right| \left([g''(x)]^q \Delta_2(k,\tau,\lambda;\alpha) + [g''(e_2)]^{qm} \Delta_3(k,\tau,\lambda;\alpha) \right)^{\frac{1}{q}} \right\},$$

where

$$\Delta_1(k,\tau,\lambda) = \int_0^1 t \left| \left(\frac{\tau}{k}+1\right)\lambda - t^{\frac{\tau}{k}} \right| \mathrm{d}t = \begin{cases} \dfrac{\frac{\tau}{k}\left[\left(\frac{\tau}{k}+1\right)\lambda\right]^{1+\frac{2k}{\tau}}}{\frac{\tau}{k}+2} - \dfrac{\left(\frac{\tau}{k}+1\right)\lambda}{2} + \dfrac{1}{\frac{\tau}{k}+2}, & 0 \leq \lambda \leq \dfrac{1}{\frac{\tau}{k}+1}, \\[2ex] \dfrac{\left(\frac{\tau}{k}+1\right)\lambda}{2} - \dfrac{1}{\frac{\tau}{k}+2}, & \dfrac{1}{\frac{\tau}{k}+1} < \lambda \leq 1, \end{cases} \tag{5}$$

$$\Delta_2(k,\tau,\lambda;\alpha) = \int_0^1 t \left| \left(\frac{\tau}{k}+1\right)\lambda - t^{\frac{\tau}{k}} \right| t^{\alpha} \mathrm{d}t = \begin{cases} \dfrac{\frac{2\tau}{k}\left[\left(\frac{\tau}{k}+1\right)\lambda\right]^{1+\frac{k(\alpha+2)}{\tau}}}{(\alpha+2)\left(\frac{\tau}{k}+\alpha+2\right)} - \dfrac{\left(\frac{\tau}{k}+1\right)\lambda}{\alpha+2} + \dfrac{1}{\frac{\tau}{k}+\alpha+2}, & 0 \leq \lambda \leq \dfrac{1}{\frac{\tau}{k}+1}, \\[2ex] \dfrac{\left(\frac{\tau}{k}+1\right)\lambda}{\alpha+2} - \dfrac{1}{\frac{\tau}{k}+\alpha+2}, & \dfrac{1}{\frac{\tau}{k}+1} < \lambda \leq 1, \end{cases} \tag{6}$$

and

$$\Delta_3(k,\tau,\lambda;\alpha) = \int_0^1 t \left| \left(\frac{\tau}{k}+1\right)\lambda - t^{\frac{\tau}{k}} \right| \left(1 - t^{\alpha}\right) \mathrm{d}t = \Delta_1(k,\tau,\lambda) - \Delta_2(k,\tau,\lambda;\alpha), \quad 0 \leq \lambda \leq 1. \tag{7}$$

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Proof. Suppose that $q=1$. From Lemma 1, we have

$$\begin{aligned}\left|\mathrm{L}_{g,\eta}(\tau,k;x,\lambda,m,e_1,e_2)\right| \leq& \left|\frac{\eta^{\frac{\tau}{k}+2}(x,e_1,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)}\right|\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda-t^{\frac{\tau}{k}}\right|\left|g''\left(me_1+t\eta(x,e_1,m)\right)\right|\mathrm{d}t \\ &+\left|\frac{(-1)^{\frac{\tau}{k}+2}\eta^{\frac{\tau}{k}+2}(x,e_2,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)}\right|\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda-t^{\frac{\tau}{k}}\right|\left|g''\left(me_2+t\eta(x,e_2,m)\right)\right|\mathrm{d}t.\end{aligned} \tag{8}$$

Using the generalised logarithmically (α,m)-preinvexity of g'' and employing the inequality of $u^s\leq(u-1)s+1$ for all $0\leq s\leq 1$ with $u>0$, we know that for any $t\in[0,1]$,

$$\begin{aligned}&\left|g''\left(me_1+t\eta(x,e_1,m)\right)\right| \\ &\leq\left[g''(e_1)\right]^{m(1-t^\alpha)}\left[g''(x)\right]^{t^\alpha}=\left[g''(e_1)\right]^m\left(\frac{g''(x)}{[g''(e_1)]^m}\right)^{t^\alpha} \\ &\leq\left[g''(e_1)\right]^m\left[\left(\frac{g''(x)}{[g''(e_1)]^m}-1\right)t^\alpha+1\right]=g''(x)t^\alpha+\left[g''(e_1)\right]^m(1-t^\alpha)\end{aligned} \tag{9}$$

and

$$\left|g''\left(me_2+t\eta(x,e_2,m)\right)\right|\leq g''(x)t^\alpha+\left[g''(e_2)\right]^m(1-t^\alpha). \tag{10}$$

Using (9) and (10) in (8), we have

$$\begin{aligned}\left|\mathrm{L}_{g,\eta}(\tau,k;x,\lambda,m,e_1,e_2)\right| \leq& \left|\frac{\eta^{\frac{\tau}{k}+2}(x,e_1,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)}\right|\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda-t^{\frac{\tau}{k}}\right|\left(g''(x)t^\alpha+\left[g''(e_1)\right]^m(1-t^\alpha)\right)\mathrm{d}t \\ &+\left|\frac{(-1)^{\frac{\tau}{k}+2}\eta^{\frac{\tau}{k}+2}(x,e_2,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)}\right|\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda-t^{\frac{\tau}{k}}\right|\left(g''(x)t^\alpha+\left[g''(e_2)\right]^m(1-t^\alpha)\right)\mathrm{d}t \\ =& \left|\frac{\eta^{\frac{\tau}{k}+2}(x,e_1,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)}\right|\left[g''(x)\Delta_2(k,\tau,\lambda;\alpha)+\left[g''(e_1)\right]^m\Delta_3(k,\tau,\lambda;\alpha)\right] \\ &+\left|\frac{(-1)^{\frac{\tau}{k}+2}\eta^{\frac{\tau}{k}+2}(x,e_2,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)}\right|\left[g''(x)\Delta_2(k,\tau,\lambda;\alpha)+\left[g''(e_2)\right]^m\Delta_3(k,\tau,\lambda;\alpha)\right],\end{aligned}$$

which completes the proof for this case. Using the power mean inequality for $q>1$, we have

$$\begin{aligned}&\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda-t^{\frac{\tau}{k}}\right|\left|g''\left(me_1+t\eta(x,e_1,m)\right)\right|\mathrm{d}t \\ &\leq\left(\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda-t^{\frac{\tau}{k}}\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda-t^{\frac{\tau}{k}}\right|\left|g''\left(me_1+t\eta(x,e_1,m)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\end{aligned} \tag{11}$$

and

$$\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda - t^{\frac{\tau}{k}}\right|\left|g''\left(me_2+t\eta(x,e_2,m)\right)\right|\mathrm{d}t$$

$$\leq\left(\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda - t^{\frac{\tau}{k}}\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda - t^{\frac{\tau}{k}}\right|\left|g''\left(me_2+t\eta(x,e_2,m)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}. \tag{12}$$

Using the generalised logarithmically (α,m)-preinvexity of $\left(g''\right)^q$, and employing the inequality of $u^s\leq(u-1)s+1$ for all $0\leq s\leq 1$ with $u>0$, we know that for any $t\in[0,1]$,

$$\left|g''\left(me_1+t\eta(x,e_1,m)\right)\right|^q\leq\left[g''(x)\right]^q t^\alpha+\left[g''(e_1)\right]^{qm}\left(1-t^\alpha\right) \tag{13}$$

and

$$\left|g''\left(me_2+t\eta(x,e_2,m)\right)\right|^q\leq\left[g''(x)\right]^q t^\alpha+\left[g''(e_2)\right]^{qm}\left(1-t^\alpha\right). \tag{14}$$

Using (8) and (11)-(14), we have

$$\left|\mathrm{L}_{g,\eta}(\tau,k;x,\lambda,m,e_1,e_2)\right|$$

$$\leq\Delta_1^{1-\frac{1}{q}}(k,\tau,\lambda)\left\{\left|\frac{\eta^{\frac{\tau}{k}+2}(x,e_1,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)}\right|\left(\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda-t^{\frac{\tau}{k}}\right|\left\{\left[g''(x)\right]^q t^\alpha+\left[g''(e_1)\right]^{qm}\left(1-t^\alpha\right)\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right.$$

$$\left.+\left|\frac{(-1)^{\frac{\tau}{k}+2}\eta^{\frac{\tau}{k}+2}(x,e_2,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)}\right|\left(\int_0^1 t\left|\left(\frac{\tau}{k}+1\right)\lambda-t^{\frac{\tau}{k}}\right|\left\{\left[g''(x)\right]^q t^\alpha+\left[g''(e_2)\right]^{qm}\left(1-t^\alpha\right)\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right\}$$

$$=\Delta_1^{1-\frac{1}{q}}(k,\tau,\lambda)\left\{\left|\frac{\eta^{\frac{\tau}{k}+2}(x,e_1,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)}\right|\left(\left[g''(x)\right]^q\Delta_2(k,\tau,\lambda;\alpha)+\left[g''(e_1)\right]^{qm}\Delta_3(k,\tau,\lambda;\alpha)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left|\frac{(-1)^{\frac{\tau}{k}+2}\eta^{\frac{\tau}{k}+2}(x,e_2,m)}{\left(\frac{\tau}{k}+1\right)\eta(e_2,e_1,m)}\right|\left(\left[g''(x)\right]^q\Delta_2(k,\tau,\lambda;\alpha)+\left[g''(e_2)\right]^{qm}\Delta_3(k,\tau,\lambda;\alpha)\right)^{\frac{1}{q}}\right\},$$

which completes the proof.

Corollary 1. In Theorem 3 if we put the mapping $\eta(e_2,e_1,m)=e_2-me_1$ with $m=1$ and take $x=\dfrac{e_1+e_2}{2}$ for some fixed $\alpha\in(0,1]$, then we have the following inequality:

$$\left|\frac{2^{\frac{\tau}{k}-1}}{(e_2-e_1)^{\frac{\tau}{k}-1}}\mathrm{L}_g\left(\tau,k;\frac{e_1+e_2}{2},\lambda,1,e_1,e_2\right)\right|$$

$$=\left|(1-\lambda)g\left(\frac{e_1+e_2}{2}\right)+\lambda\left[\frac{g(e_1)+g(e_2)}{2}\right]-\frac{2^{\frac{\tau}{k}-1}\Gamma_k(\tau+k)}{(e_2-e_1)^{\frac{\tau}{k}}}\left[{}_k\mathrm{J}^{\tau}_{\left(\frac{e_1+e_2}{2}\right)^-}g(e_1)+{}_k\mathrm{J}^{\tau}_{\left(\frac{e_1+e_2}{2}\right)^+}g(e_2)\right]\right|$$

$$\leq \frac{(e_2-e_1)^2}{8\left(\frac{\tau}{k}+1\right)}\Delta_1^{1-\frac{1}{q}}(k,\tau,\lambda)\left\{\left(\left[g''\left(\frac{e_1+e_2}{2}\right)\right]^q \Delta_2(k,\tau,\lambda;\alpha)+\left[g''(e_1)\right]^q \Delta_3(k,\tau,\lambda;\alpha)\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\left[g''\left(\frac{e_1+e_2}{2}\right)\right]^q \Delta_2(k,\tau,\lambda;\alpha)+\left[g''(e_2)\right]^q \Delta_3(k,\tau,\lambda;\alpha)\right)^{\frac{1}{q}}\right\}.$$

Theorem 4. If $\left(g''\right)^q$ is generalised logarithmically (α,m)-preinvex on A for some fixed $\alpha,m\in(0,1]$ and $q>1$, then the following inequality for k-fractional integrals with $x\in[e_1,e_2]$, $\lambda\in[0,1]$, $\tau>0$, $k>0$ holds:

$$|\mathrm{L}_{g,\eta}(\tau,k;x,\lambda,m,e_1,e_2)|\leq \Delta_4^{\frac{1}{p}}(k,\tau,\lambda,p)\left\{\left|\frac{\eta^{\frac{\tau}{k}+2}(x,e_1,m)}{(\frac{\tau}{k}+1)\eta(e_2,e_1,m)}\right|\left(\left[g''(x)\right]^q\frac{1}{\alpha+1}+\left[g''(e_1)\right]^{qm}\frac{\alpha}{\alpha+1}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left|\frac{(-1)^{\frac{\tau}{k}+2}\eta^{\frac{\tau}{k}+2}(x,e_2,m)}{(\frac{\tau}{k}+1)\eta(e_2,e_1,m)}\right|\left(\left[g''(x)\right]^q\frac{1}{\alpha+1}+\left[g''(e_2)\right]^{qm}\frac{\alpha}{\alpha+1}\right)^{\frac{1}{q}}\right\},$$

where $p^{-1}+q^{-1}=1$,

$$\Delta_4(k,\tau,\lambda,p)=\int_0^1 t^p\left|\left(\frac{\tau}{k}+1\right)\lambda-t^{\frac{\tau}{k}}\right|^p\mathrm{d}t$$

$$=\begin{cases}\dfrac{1}{p\left(\frac{\tau}{k}+1\right)+1}, & \lambda=0,\\[2ex] \left[\dfrac{k\left[\left(\frac{\tau}{k}+1\right)\lambda\right]^{\frac{k+kp\left(\frac{\tau}{k}+1\right)}{\tau}}}{\tau}\beta\left(\dfrac{k(1+p)}{\tau},1+p\right)+\dfrac{k\left[1-\left(\frac{\tau}{k}+1\right)\lambda\right]^{p+1}}{\tau(p+1)}\,{}_2F_1\left(1-\dfrac{k(1+p)}{\tau},1;p+2;1-\left(\frac{\tau}{k}+1\right)\lambda\right)\right], & 0<\lambda\leq\dfrac{1}{\frac{\tau}{k}+1},\\[2ex] \dfrac{k\left[\left(\frac{\tau}{k}+1\right)\lambda\right]^{\frac{k+kp\left(\frac{\tau}{k}+1\right)}{\tau}}}{\tau}\beta\left(\dfrac{1}{\left(\frac{\tau}{k}+1\right)\lambda};\dfrac{k(1+p)}{\tau},1+p\right), & \dfrac{1}{\frac{\tau}{k}+1}<\lambda\leq 1,\end{cases}$$

with $\beta(x,y)=\dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,\ x,y>0,$

$$\beta(a;x,y)=\int_0^a t^{x-1}(1-t)^{y-1}\mathrm{d}t,\ 0<a<1,\ x,y>0,$$

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$$ {}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t, \ c > b > 0, |z| < 1. $$

Proof. Continuing from inequality (8) and the Hölder's inequality, we have

$$ \begin{aligned} &\left|\mathrm{L}_{g,\eta}(\tau,k;x,\lambda,m,e_1,e_2)\right| \\ &\le \left|\frac{\eta^{\frac{\tau}{k}+2}(x,e_1,m)}{(\frac{\tau}{k}+1)\eta(e_2,e_1,m)}\right| \left[\int_0^1 t^p \left|\left(\frac{\tau}{k}+1\right)\lambda - t^{\frac{\tau}{k}}\right|^p \mathrm{d}t\right]^{\frac{1}{p}} \left[\int_0^1 \left|g''\left(me_1 + t\eta(x,e_1,m)\right)\right|^q \mathrm{d}t\right]^{\frac{1}{q}} \\ &+ \left|\frac{(-1)^{\frac{\tau}{k}+2}\eta^{\frac{\tau}{k}+2}(x,e_2,m)}{(\frac{\tau}{k}+1)\eta(e_2,e_1,m)}\right| \left[\int_0^1 t^p \left|\left(\frac{\tau}{k}+1\right)\lambda - t^{\frac{\tau}{k}}\right|^p \mathrm{d}t\right]^{\frac{1}{p}} \left[\int_0^1 \left|g''\left(me_2 + t\eta(x,e_2,m)\right)\right|^q \mathrm{d}t\right]^{\frac{1}{q}}. \end{aligned} \tag{15} $$

Using the generalised logarithmically (α,m)-preinvexity of $(g'')^q$ and employing the inequality of $u^s \le (u-1)s+1$ for all $0 \le s \le 1$ with $u > 0$, we know that for any $t \in [0,1]$,

$$ \begin{aligned} \int_0^1 \left|g''\left(me_1 + t\eta(x,e_1,m)\right)\right|^q \mathrm{d}t &\le \int_0^1 \left[[g''(e_1)]^{qm(1-t^\alpha)}[g''(x)]^{qt^\alpha}\right]\mathrm{d}t \\ &= [g''(e_1)]^{qm}\int_0^1 \left(\frac{g''(x)}{[g''(e_1)]^m}\right)^{qt^\alpha}\mathrm{d}t \\ &\le [g''(e_1)]^{qm}\int_0^1 \left\{\left[\left(\frac{g''(x)}{[g''(e_1)]^m}\right)^q - 1\right]t^\alpha + 1\right\}\mathrm{d}t \\ &= [g''(x)]^q\frac{1}{\alpha+1} + [g''(e_1)]^{qm}\frac{\alpha}{\alpha+1} \end{aligned} \tag{16} $$

and

$$ \int_0^1 \left|g''\left(me_2 + t\eta(x,e_2,m)\right)\right|^q \mathrm{d}t \le [g''(x)]^q\frac{1}{\alpha+1} + [g''(e_2)]^{qm}\frac{\alpha}{\alpha+1}. \tag{17} $$

Hence if we use (16) and (17) in (15), then we obtain the desired result. This ends the proof.

PRODUCT-TYPE INEQUALITIES THROUGH GENERALISED LOGARITHMICALLY (α,m)-PREINVEXITY

Next, we establish several Hadamard-type integral inequalities for the product of generalised logarithmically (α,m)-preinvex functions and other convex functions.

Theorem 5. Let $e_1, e_2 \in [0,+\infty), e_1 < e_2$ with $\eta(e_2,e_1,m) > 0$ for some fixed $m \in (0,1]$, and let $f, g : [me_1, me_1 + \eta(e_2,e_1,m)] \to (0,\infty)$ be non-negative integrable. If f is generalised quasi-m-preinvex on $\left[me_1, me_1 + \eta(e_2,e_1,m)\right]$ and g^q is generalised logarithmically (α,m)-preinvex on $\left[me_1, me_1 + \eta(e_2,e_1,m)\right]$ for some fixed $\alpha, m \in (0,1]$ with $q \ge 1$, then we have

$$ \frac{1}{\eta(e_2,e_1,m)} \int_{me_1}^{me_1+\eta(e_2,e_1,m)} f(x)g(x)\mathrm{d}x \le \max\{f(e_1), f(e_2)\} \left\{[g(e_1)]^{qm}\frac{\alpha}{\alpha+1} + [g(e_2)]^q\frac{1}{\alpha+1}\right\}^{\frac{1}{q}}. $$

Proof. Taking $x = me_1 + t\eta(e_2, e_1, m)$ for $t \in [0,1]$, using the Hölder's inequality, and employing the conditions that f is generalised quasi-m-preinvex and g^q is generalised logarithmically (α, m)-preinvex, the following is figured out:

$$\begin{aligned}
&\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)} f(x)g(x)\mathrm{d}x \\
&= \int_0^1 f\left[me_1 + t\eta(e_2,e_1,m)\right] g\left[me_1 + t\eta(e_2,e_1,m)\right]\mathrm{d}t \\
&\leq \left\{\int_0^1 f\left[me_1 + t\eta(e_2,e_1,m)\right]\mathrm{d}t\right\}^{1-\frac{1}{q}} \left\{\int_0^1 f\left[me_1 + t\eta(e_2,e_1,m)\right] g^q\left[me_1 + t\eta(e_2,e_1,m)\right]\mathrm{d}t\right\}^{\frac{1}{q}} \qquad (18) \\
&\leq \left[\max\left\{f(e_1), f(e_2)\right\}\right]^{1-\frac{1}{q}} \left\{\int_0^1 \max\left\{f(e_1), f(e_2)\right\}\left[g(e_1)\right]^{qm(1-t^\alpha)}\left[g(e_2)\right]^{qt^\alpha}\mathrm{d}t\right\}^{\frac{1}{q}} \\
&= \max\left\{f(e_1), f(e_2)\right\}\left\{\int_0^1 \left[g(e_1)\right]^{qm(1-t^\alpha)}\left[g(e_2)\right]^{qt^\alpha}\mathrm{d}t\right\}^{\frac{1}{q}}.
\end{aligned}$$

Employing the inequality $u^s \leq (u-1)s + 1$ for all $0 \leq s \leq 1$ with $u > 0$, we obtain

$$\begin{aligned}
\int_0^1 \left[\left[g(e_1)\right]^{qm(1-t^\alpha)}\left[g(e_2)\right]^{qt^\alpha}\right]\mathrm{d}t &= \left[g(e_1)\right]^{qm}\int_0^1 \left(\frac{g(e_2)}{[g(e_1)]^m}\right)^{qt^\alpha}\mathrm{d}t \\
&\leq \left[g(e_1)\right]^{qm}\int_0^1 \left\{\left[\left(\frac{g(e_2)}{[g(e_1)]^m}\right)^q - 1\right]t^\alpha + 1\right\}\mathrm{d}t \qquad (19) \\
&= \left[g(e_1)\right]^{qm}\frac{\alpha}{\alpha+1} + [g(e_2)]^q\frac{1}{\alpha+1}.
\end{aligned}$$

Hence if we use (19) in (18), we obtain the desired result. This completes the proof.

Theorem 6. Let $g_i : [me_1, me_1 + \eta(e_2, e_1, m)] \to (0, \infty)$ be generalised logarithmically (α_i, m)-preinvex on $\mathrm{Int}\left[me_1, me_1 + \eta(e_2, e_1, m)\right]$ with some fixed $\alpha_i, m \in (0,1]$ $(i = 1, 2, \cdots, n)$, $e_1, e_2 \in \mathrm{Int}\left[me_1, me_1 + \eta(e_2, e_1, m)\right]$, $0 < e_1 < e_2$ and $\eta(e_2, e_1, m) > 0$. If $\sum_{i=1}^n \beta_i = 1$, $\beta_i > 0$, then the following inequality holds:

$$\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)} \prod_{i=1}^n g_i(x)\mathrm{d}x \leq \sum_{i=1}^n \left(\left[g_i(e_1)\right]^{\frac{m}{\beta_i}}\frac{\alpha_i\beta_i}{\alpha_i+1} + \left[g_i(e_2)\right]^{\frac{1}{\beta_i}}\frac{\beta_i}{\alpha_i+1}\right).$$

Proof. Using the inequality

$$g_1 \cdot g_2 \cdots g_n \leq \beta_1(g_1)^{\frac{1}{\beta_1}} + \beta_2(g_2)^{\frac{1}{\beta_2}} + \cdots + \beta_n(g_n)^{\frac{1}{\beta_n}}, \qquad \beta_i > 0,\ \sum_{i=1}^n \beta_i = 1,$$

and using the generalised logarithmically (α_i, m)-preinvexity of g_i, we have

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$$
\begin{aligned}
\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)}\prod_{i=1}^{n}g_i(x)\mathrm{d}x &= \int_0^1\prod_{i=1}^{n}g_i\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t\\
&\le \int_0^1\sum_{i=1}^{n}\beta_i\left(g_i\left[me_1+t\eta(e_2,e_1,m)\right]\right)^{\frac{1}{\beta_i}}\mathrm{d}t\\
&\le \int_0^1\sum_{i=1}^{n}\beta_i\left[g_i(e_1)\right]^{\frac{1}{\beta_i}m(1-t^{\alpha_i})}\left[g_i(e_2)\right]^{\frac{1}{\beta_i}t^{\alpha_i}}\mathrm{d}t\\
&= \sum_{i=1}^{n}\left(\beta_i\int_0^1\left[g_i(e_1)\right]^{\frac{1}{\beta_i}m(1-t^{\alpha_i})}\left[g_i(e_2)\right]^{\frac{1}{\beta_i}t^{\alpha_i}}\mathrm{d}t\right).
\end{aligned}
\tag{20}
$$

Using the fact that $u^s \le (u-1)s+1$ for all $0\le s\le 1$ with $u>0$, we have

$$
\begin{aligned}
\int_0^1\left[g_i(e_1)\right]^{\frac{1}{\beta_i}m(1-t^{\alpha_i})}\left[g_i(e_2)\right]^{\frac{1}{\beta_i}t^{\alpha_i}}\mathrm{d}t &= \left[g_i(e_1)\right]^{\frac{m}{\beta_i}}\int_0^1\left(\frac{g_i(e_2)}{[g_i(e_1)]^m}\right)^{\frac{1}{\beta_i}t^{\alpha_i}}\mathrm{d}t\\
&\le \left[g_i(e_1)\right]^{\frac{m}{\beta_i}}\int_0^1\left\{\left[\left(\frac{g_i(e_2)}{[g_i(e_1)]^m}\right)^{\frac{1}{\beta_i}}-1\right]t^{\alpha_i}+1\right\}\mathrm{d}t\\
&= \left[g_i(e_1)\right]^{\frac{m}{\beta_i}}\frac{\alpha_i}{\alpha_i+1}+\left[g_i(e_2)\right]^{\frac{1}{\beta_i}}\frac{1}{\alpha_i+1}.
\end{aligned}
\tag{21}
$$

Hence if we use (21) in (20), we obtain the desired result. This completes the proof.

Theorem 7. Let $e_1,e_2\in[0,+\infty), e_1<e_2$ with $\eta(e_2,e_1,m)>0$ for some fixed $m\in(0,1]$, and let $f,g:[me_1,me_1+\eta(e_2,e_1,m)]\to(0,\infty)$ be non-negative integrable. If f is generalised (m,s)-Breckner-preinvex on $\left[me_1,me_1+\eta(e_2,e_1,m)\right]$ for some fixed $s,m\in(0,1]$, and g^q is generalised logarithmically (α,m)-preinvex on $\left[me_1,me_1+\eta(e_2,e_1,m)\right]$ for some fixed $\alpha,m\in(0,1]$ with $q\ge 1$, then the following inequality holds:

$$
\begin{aligned}
&\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)}f(x)g(x)\mathrm{d}x\\
&\le\left\{\frac{mf(e_1)+f(e_2)}{s+1}\right\}^{1-\frac{1}{q}}\left\{mf(e_1)\left(\left(g^q(e_2)-g^{qm}(e_1)\right)\beta(\alpha+1,s+1)+\frac{g^{qm}(e_1)}{s+1}\right)\right.\\
&\quad\left.+f(e_2)\left(\frac{g^q(e_2)-g^{qm}(e_1)}{s+\alpha+1}+\frac{g^{qm}(e_1)}{s+1}\right)\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Proof. Taking $x=me_1+t\eta(e_2,e_1,m)$ for $t\in[0,1]$, using the power mean inequality, and employing the conditions that f is generalised (m,s)-Breckner-preinvex and g^q is generalised logarithmically (α,m)-preinvex, the following is figured out:

$$
\begin{aligned}
&\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)}f(x)g(x)\mathrm{d}x\\
&=\int_0^1 f\left[me_1+t\eta(e_2,e_1,m)\right]g\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t\\
&\le\left\{\int_0^1 f\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t\right\}^{1-\frac{1}{q}}\left\{\int_0^1 f\left[me_1+t\eta(e_2,e_1,m)\right]g^q\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t\right\}^{\frac{1}{q}}
\end{aligned}
\tag{22}
$$

$$\le \left\{\int_0^1 \left[m(1-t)^s f(e_1)+t^s f(e_2)\right]\mathrm{d}t\right\}^{1-\frac{1}{q}} \left\{\int_0^1 \left[m(1-t)^s f(e_1)+t^s f(e_2)\right]\left[g(e_1)\right]^{qm(1-t^{\alpha})}\left[g(e_2)\right]^{qt^{\alpha}}\,\mathrm{d}t\right\}^{\frac{1}{q}}$$

$$= g^m(e_1)\left\{\frac{mf(e_1)+f(e_2)}{s+1}\right\}^{1-\frac{1}{q}} \left\{mf(e_1)\int_0^1 (1-t)^s\left[\frac{g(e_2)}{[g(e_1)]^m}\right]^{qt^{\alpha}}\mathrm{d}t + f(e_2)\int_0^1 t^s \left[\frac{g(e_2)}{[g(e_1)]^m}\right]^{qt^{\alpha}}\mathrm{d}t\right\}^{\frac{1}{q}}.$$

Here, letting $v=\dfrac{g(e_2)}{[g(e_1)]^m}$ and using the inequality $v^{qt^{\alpha}} \le (v^q-1)t^{\alpha}+1$ for all $0\le t\le 1$, we have

$$\int_0^1 (1-t)^s v^{qt^{\alpha}}\mathrm{d}t \le \int_0^1 (1-t)^s\left[(v^q-1)t^{\alpha}+1\right]\mathrm{d}t = (v^q-1)\beta(\alpha+1,s+1)+\frac{1}{s+1} \tag{23}$$

and

$$\int_0^1 t^s v^{qt^{\alpha}}\mathrm{d}t \le \int_0^1 t^s\left[(v^q-1)t^{\alpha}+1\right]\mathrm{d}t = \frac{v^q-1}{s+\alpha+1}+\frac{1}{s+1}. \tag{24}$$

Hence if we use (23) and (24) in (22), we obtain the desired result. This ends the proof.

Theorem 8. Let $e_1,e_2\in[0,+\infty), e_1<e_2$ with $\eta(e_2,e_1,m)>0$ for some fixed $m\in(0,1]$, and let $f,g:[me_1,me_1+\eta(e_2,e_1,m)]\to(0,\infty)$ be non-negative integrable, either increasing or decreasing synchronously. If f, g are generalised logarithmically (α_1,m), (α_2,m)-preinvex mappings on $\mathrm{Int}\left[me_1,me_1+\eta(e_2,e_1,m)\right]$ with some fixed $\alpha_1,\alpha_2\in(0,1]$, then we have

$$\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)} f(x)\mathrm{d}x\cdot \mathrm{L}_2\left[g(e_1),g(e_2)\right] + \frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)} g(x)\mathrm{d}x\cdot \mathrm{L}_1\left[f(e_1),f(e_2)\right]$$

$$\le \frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)} f(x)g(x)\mathrm{d}x + \left\{\mathrm{L}_1\left[f^2(e_1),f^2(e_2)\right]\mathrm{L}_2\left[g^2(e_1),g^2(e_2)\right]\right\}^{\frac{1}{2}},$$

where

$$\mathrm{L}_i(\phi,\varphi)=\int_0^1 \phi^{m(1-t^{\alpha_i})}\varphi^{t^{\alpha_i}}\mathrm{d}t = \phi^m\int_0^1\left[\frac{\varphi}{\phi^m}\right]^{t^{\alpha_i}}\mathrm{d}t, \qquad (i=1,2).$$

Proof. Since f, g are generalised logarithmically $(\alpha_1,m),(\alpha_2,m)$-preinvex, we have

$$f[me_1+t\eta(e_2,e_1,m)] \le \left[f(e_1)\right]^{m(1-t^{\alpha_1})}\left[f(e_2)\right]^{t^{\alpha_1}},$$

$$g[me_1+t\eta(e_2,e_1,m)] \le \left[g(e_1)\right]^{m(1-t^{\alpha_2})}\left[g(e_2)\right]^{t^{\alpha_2}}.$$

Now, using $\langle x_1-x_2,x_3-x_4\rangle\ge 0, \left(x_1,x_2,x_3,x_4\in\mathrm{R}\right)$, $x_1<x_2$ and $x_3<x_4$, one has

$$f\left[me_1+t\eta(e_2,e_1,m)\right]\left[g(e_1)\right]^{m(1-t^{\alpha_2})}\left[g(e_2)\right]^{t^{\alpha_2}} + g\left[me_1+t\eta(e_2,e_1,m)\right]\left[f(e_1)\right]^{m(1-t^{\alpha_1})}\left[f(e_2)\right]^{t^{\alpha_1}},$$

$$\le f\left[me_1+t\eta(e_2,e_1,m)\right]g\left[me_1+t\eta(e_2,e_1,m)\right] + \left[f(e_1)\right]^{m(1-t^{\alpha_1})}\left[f(e_2)\right]^{t^{\alpha_1}}\left[g(e_1)\right]^{m(1-t^{\alpha_2})}\left[g(e_2)\right]^{t^{\alpha_2}}.$$

Integrating the inequalities above with respect to t on $[0,1]$, we have

$$\int_0^1 f\left[me_1+t\eta(e_2,e_1,m)\right]\left[g(e_1)\right]^{m(1-t^{\alpha_2})}\left[g(e_2)\right]^{t^{\alpha_2}}\mathrm{d}t + \int_0^1 g\left[me_1+t\eta(e_2,e_1,m)\right]\left[f(e_1)\right]^{m(1-t^{\alpha_1})}\left[f(e_2)\right]^{t^{\alpha_1}}\mathrm{d}t$$

$$\le \int_0^1 f\left[me_1+t\eta(e_2,e_1,m)\right]g\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t + \int_0^1 \left[f(e_1)\right]^{m(1-t^{\alpha_1})}\left[f(e_2)\right]^{t^{\alpha_1}}\left[g(e_1)\right]^{m(1-t^{\alpha_2})}\left[g(e_2)\right]^{t^{\alpha_2}}\mathrm{d}t.$$

Using the property that f,g are either increasing or decreasing synchronously, and then employing the Cauchy inequality, we have

$$\int_0^1 f\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t\int_0^1\left[g(e_1)\right]^{m(1-t^{\alpha_2})}\left[g(e_2)\right]^{t^{\alpha_2}}\mathrm{d}t$$
$$+\int_0^1 g\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t\int_0^1\left[f(e_1)\right]^{m(1-t^{\alpha_1})}\left[f(e_2)\right]^{t^{\alpha_1}}\mathrm{d}t$$
$$\le\int_0^1 f\left[me_1+t\eta(e_2,e_1,m)\right]g\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t$$
$$+\int_0^1\left(\left[f(e_1)\right]^{m(1-t^{\alpha_1})}\left[f(e_2)\right]^{t^{\alpha_1}}\right)\left(\left[g(e_1)\right]^{m(1-t^{\alpha_2})}\left[g(e_2)\right]^{t^{\alpha_2}}\right)\mathrm{d}t$$
$$\le\int_0^1 f\left[me_1+t\eta(e_2,e_1,m)\right]g\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t$$
$$+\left\{\int_0^1\left[f(e_1)\right]^{2m(1-t^{\alpha_1})}\left[f(e_2)\right]^{2t^{\alpha_1}}\mathrm{d}t\int_0^1\left[g(e_1)\right]^{2m(1-t^{\alpha_2})}\left[g(e_2)\right]^{2t^{\alpha_2}}\mathrm{d}t\right\}^{\frac{1}{2}}.$$

Taking $x=me_1+t\eta(e_2,e_1,m)$ for $t\in[0,1]$, we obtain the desired result. This completes the proof.

Corollary 2. In Theorem 8 if we put the mapping $\eta(e_2,e_1,m)=e_2-me_1$ with $m=1$, and choose $\alpha_1=\alpha_2=1$, then we have the following inequality for logarithmically convex functions:

$$\frac{g(e_2)-g(e_1)}{\ln g(e_2)-\ln g(e_1)}\int_{e_1}^{e_2}f(x)\mathrm{d}x+\frac{f(e_2)-f(e_1)}{\ln f(e_2)-\ln f(e_1)}\int_{e_1}^{e_2}g(x)\mathrm{d}x$$
$$\le\int_{e_1}^{e_2}f(x)g(x)\mathrm{d}x+\frac{(e_2-e_1)\left[f(e_2)g(e_2)-f(e_1)g(e_1)\right]}{\ln\left[f(e_2)g(e_2)\right]-\ln\left[f(e_1)g(e_1)\right]}$$
$$\le\int_{e_1}^{e_2}f(x)g(x)\mathrm{d}x+\frac{e_2-e_1}{2}\left\{\frac{f^2(e_2)-f^2(e_1)}{\left(\ln f(e_2)-\ln f(e_1)\right)}\frac{g^2(e_2)-g^2(e_1)}{\left(\ln g(e_2)-\ln g(e_1)\right)}\right\}^{\frac{1}{2}}.$$

Theorem 9. Under the assumptions of Theorem 8, we have

$$\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)}f(x)\mathrm{d}x\cdot\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)}g(x)\mathrm{d}x$$
$$\le\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)}f(x)g(x)\mathrm{d}x$$
$$\le Q_1\left[f(e_1)g(e_1)\right]^m+Q_2\left[f(e_1)\right]^m g(e_2)+Q_3 f(e_2)\left[g(e_1)\right]^m+Q_4 f(e_2)g(e_2),$$

where

$$Q_1=\frac{\alpha_1\alpha_2(\alpha_1+\alpha_2+2)}{(\alpha_1+\alpha_2+1)(\alpha_1+1)(\alpha_2+1)},\quad Q_2=\frac{\alpha_1}{(\alpha_1+\alpha_2+1)(\alpha_2+1)},$$
$$Q_3=\frac{\alpha_2}{(\alpha_1+\alpha_2+1)(\alpha_1+1)},\quad Q_4=\frac{1}{\alpha_1+\alpha_2+1}.$$

Proof. Since $f,g:[me_1,me_1+\eta(e_2,e_1,m)]\to(0,\infty)$ are generalised logarithmically $(\alpha_1,m),(\alpha_2,m)$ -preinvex, we have

$$f\left[me_1+t\eta(e_2,e_1,m)\right]\le\left[f(e_1)\right]^{m(1-t^{\alpha_1})}\left[f(e_2)\right]^{t^{\alpha_1}},$$
$$g\left[me_1+t\eta(e_2,e_1,m)\right]\le\left[g(e_1)\right]^{m(1-t^{\alpha_2})}\left[g(e_2)\right]^{t^{\alpha_2}}.$$

Multiplying the above inequalities on either side (i.e. from left to left and right to right), and using the inequality of $u^s\le(u-1)s+1$ for all $0\le s\le 1$ with $u>0$, we obtain

$$\begin{aligned}
&f\left[me_1+t\eta(e_2,e_1,m)\right]g\left[me_1+t\eta(e_2,e_1,m)\right]\\
&\le\left(\left[f(e_1)\right]^{m(1-t^{\alpha_1})}\left[f(e_2)\right]^{t^{\alpha_1}}\right)\left(\left[g(e_1)\right]^{m(1-t^{\alpha_2})}\left[g(e_2)\right]^{t^{\alpha_2}}\right)\\
&=\left[f(e_1)g(e_1)\right]^m\left[\frac{f(e_2)}{[f(e_1)]^m}\right]^{t^{\alpha_1}}\left[\frac{g(e_2)}{[g(e_1)]^m}\right]^{t^{\alpha_2}}\\
&\le\left[f(e_1)g(e_1)\right]^m\left[\left(\frac{f(e_2)}{[f(e_1)]^m}-1\right)t^{\alpha_1}+1\right]\left[\left(\frac{g(e_2)}{[g(e_1)]^m}-1\right)t^{\alpha_2}+1\right]\\
&=(1-t^{\alpha_1})(1-t^{\alpha_2})\left[f(e_1)\right]^m\left[g(e_1)\right]^m+t^{\alpha_2}(1-t^{\alpha_1})\left[f(e_1)\right]^m g(e_2)\\
&\quad+t^{\alpha_1}(1-t^{\alpha_2})f(e_2)\left[g(e_1)\right]^m+t^{\alpha_1}t^{\alpha_2}f(e_2)g(e_2).
\end{aligned}\tag{25}$$

Since f, g are either increasing or decreasing synchronously, by using the following Chebyshev inequality,

$$\frac{1}{e_2-e_1}\int_{e_1}^{e_2}f(x)g(x)\mathrm{d}x\ge\frac{1}{e_2-e_1}\int_{e_1}^{e_2}f(x)\mathrm{d}x\frac{1}{e_2-e_1}\int_{e_1}^{e_2}g(x)\mathrm{d}x,$$

we have

$$\begin{aligned}
&\int_0^1 f\left[me_1+t\eta(e_2,e_1,m)\right]g\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t\\
&\ge\int_0^1 f\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t\int_0^1 g\left[me_1+t\eta(e_2,e_1,m)\right]\mathrm{d}t\\
&=\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)}f(x)\mathrm{d}x\cdot\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)}g(x)\mathrm{d}x.
\end{aligned}$$

On the other hand, integrating both sides of the inequality (25) according to t over $[0,1]$, we have

$$\begin{aligned}
&\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)}f(x)\mathrm{d}x\cdot\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)}g(x)\mathrm{d}x\\
&\le\frac{1}{\eta(e_2,e_1,m)}\int_{me_1}^{me_1+\eta(e_2,e_1,m)}f(x)g(x)\mathrm{d}x\\
&\le\left[f(e_1)g(e_1)\right]^m\int_0^1\left(1-t^{\alpha_1}\right)\left(1-t^{\alpha_2}\right)\mathrm{d}t+\left[f(e_1)\right]^m g(e_2)\int_0^1 t^{\alpha_2}\left(1-t^{\alpha_1}\right)\mathrm{d}t\\
&\quad+f(e_2)[g(e_1)]^m\int_0^1 t^{\alpha_1}\left(1-t^{\alpha_2}\right)\mathrm{d}t+f(e_2)g(e_2)\int_0^1 t^{\alpha_1}t^{\alpha_2}\mathrm{d}t\\
&=\left[f(e_1)g(e_1)\right]^m\frac{\alpha_1\alpha_2(\alpha_1+\alpha_2+2)}{(\alpha_1+\alpha_2+1)(\alpha_1+1)(\alpha_2+1)}+\left[f(e_1)\right]^m g(e_2)\frac{\alpha_1}{(\alpha_1+\alpha_2+1)(\alpha_2+1)}\\
&\quad+f(e_2)\left[g(e_1)\right]^m\frac{\alpha_2}{(\alpha_1+\alpha_2+1)(\alpha_1+1)}+f(e_2)g(e_2)\frac{1}{\alpha_1+\alpha_2+1},
\end{aligned}$$

which completes the proof.

Corollary 3. In Theorem 9 if we take $\eta(e_2,e_1,m)=e_2-me_1$, then we have the following inequality:

$$\begin{aligned}
&\frac{1}{e_2-me_1}\int_{me_1}^{e_2}f(x)\mathrm{d}x\cdot\frac{1}{e_2-me_1}\int_{me_1}^{e_2}g(x)\mathrm{d}x\\
&\le\frac{1}{e_2-me_1}\int_{me_1}^{e_2}f(x)g(x)\mathrm{d}x\\
&\le Q_1\left[f(e_1)g(e_1)\right]^m+Q_2\left[f(e_1)\right]^m g(e_2)+Q_3 f(e_2)\left[g(e_1)\right]^m+Q_4 f(e_2)g(e_2).
\end{aligned}\tag{26}$$

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APPLICATIONS TO SOME SPECIAL MEANS

For non-negative numbers e_1, e_2 $(e_1 \neq e_2)$, we define

$$A(e_1,e_2)=\frac{e_1+e_2}{2},\quad L(e_1,e_2)=\frac{e_2-e_1}{\ln e_2-\ln e_1},\quad H(e_1,e_2)=\frac{2e_1e_2}{e_1+e_2}.$$

Taking $g(x)=\frac{1}{x}$ for $x\in(0,+\infty)$, one has $[g''(x)]^q=2^q\exp(-3q\ln x)$. Note that $(-3q\ln x)''=\frac{3q}{x^2}>0$ for $q\geq 1$. This shows that $(g'')^q$ is logarithmically convex on $(0,+\infty)$.

Applying this function $g(x)$ to Corollary 1 with $k=\tau=\alpha=1$ and $q=1$, we have the following results.

Proposition 1. Let $0<e_1<e_2<+\infty$. Then we obtain the following results.

(a) For $\lambda=0$, we have

$$|A^{-1}(e_1,e_2)-L^{-1}(e_1,e_2)|\leq\frac{(e_2-e_1)^2}{16}\left[A^{-3}(e_1,e_2)+\frac{1}{3}H^{-1}\left(e_1^{\,3},e_2^{\,3}\right)\right].$$

(b) For $\lambda=1/3$, we have

$$\left|\frac{1}{3}H^{-1}(e_1,e_2)+\frac{2}{3}A^{-1}(e_1,e_2)-L^{-1}(e_1,e_2)\right|\leq\frac{(e_2-e_1)^2}{16}\left[\frac{59}{243}A^{-3}(e_1,e_2)+\frac{37}{243}H^{-1}\left(e_1^{\,3},e_2^{\,3}\right)\right].$$

(c) For $\lambda=1/2$, we have

$$\left|\frac{1}{2}H^{-1}(e_1,e_2)+\frac{1}{2}A^{-1}(e_1,e_2)-L^{-1}(e_1,e_2)\right|\leq\frac{(e_2-e_1)^2}{16}\left[\frac{1}{3}A^{-3}(e_1,e_2)+\frac{1}{3}H^{-1}\left(e_1^{\,3},e_2^{\,3}\right)\right].$$

(d) For $\lambda=1$, we have

$$\left|H^{-1}(e_1,e_2)-L^{-1}(e_1,e_2)\right|\leq\frac{(e_2-e_1)^2}{16}\left[\frac{5}{3}A^{-3}(e_1,e_2)+H^{-1}\left(e_1^{\,3},e_2^{\,3}\right)\right].$$

Proposition 2. For $0<me_1<e_2<+\infty$ with some fixed $m\in(0,1]$, we have

$$\begin{aligned}L^2\left(\exp(me_1),\exp(e_2)\right)&\leq A\left(\exp(me_1),\exp(e_2)\right)L\left(\exp(me_1),\exp(e_2)\right)\\&\leq\frac{1}{3}\left[\exp(2me_1)+\exp(me_1+e_2)+\exp(2e_2)\right].\end{aligned}$$

Proof. If we choose the logarithmically $(1,m)$-convex function $f(x)=g(x)=e^x$ for $x\in(0,\infty)$ with some fixed $m\in(0,1]$ in (26), then we obtain

$$\begin{aligned}&\frac{1}{e_2-me_1}\int_{me_1}^{e_2}e^x\mathrm{d}x\cdot\frac{1}{e_2-me_1}\int_{me_1}^{e_2}e^x\mathrm{d}x=L^2\left(\exp(me_1),\exp(e_2)\right)\\&\leq\frac{1}{e_2-me_1}\int_{me_1}^{e_2}e^{2x}\mathrm{d}x=A\left(\exp(me_1),\exp(e_2)\right)L\left(\exp(me_1),\exp(e_2)\right)\\&\leq\frac{1}{3}\left[\exp(2me_1)+\exp(me_1+e_2)+\exp(2e_2)\right].\end{aligned}$$

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CONCLUSIONS

We have established some k-fractional inequalities for a class of generalised logarithmically (α,m) -preinvex mappings. We have also proved certain Hadamard-type inequalities for products of the generalised logarithmically (α,m)-preinvex mappings with other convex mappings.

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