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Applications of higher-order derivatives to subclasses of multivalent q-starlike functions

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Abstract: Three new subfamilies of multivalent (or p -valent) q-starlike functions with respect to higher-order q-derivatives are introduced. Several properties of such families of q-starlike functions with negative coefficients including distortion theorems and radius problems are derived. For the motivation and validity of the results which are presented here, relevant connections of our findings with those that were given in earlier work are also pointed out. Finally, we have chosen to reiterate the well-demonstrated fact that any attempt to produce the rather straightforward (p,q) -variations of the results, which we have presented in this article, will be a rather trivial and inconsequential exercise, simply because the additional parameter p is obviously redundant.

Keywords: multivalent (or p -valent) functions, distortion theorems, radius problems, q-derivative operator, q-starlike functions

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INTRODUCTION

Calculus without the notion of limit is called quantum calculus, usually the so-called q-calculus or q-theory. Here the letter q apparently stands for quantum. Also, quantum calculus is not the same as quantum physics. Due to the applications in various fields of mathematics and physics, the study of q-calculus has been very attractive for many researchers. The first mathematician who gave the idea of q-calculus was Jackson [1, 2]. Then by using the concept of q-calculus, some remarkable developments have been made [3-5].

In Geometric Function Theory several subclasses belonging to the class $\mathcal{A}$ of normalised analytic functions $\mathcal{A}$ have already been investigated in different aspects [6, 7]. Ismail et al. [8] were the first who employed the q-derivative operator D_q to study a q-analogue of the class $\mathcal{S}^*$ of starlike functions in U (see Definition 6 below). In fact, historically speaking, a remarkably significant usage of the q-calculus in the context of Geometric Function Theory of Complex Analysis was basically furnished and the basic (or q-) hypergeometric functions were first used in Geometric Function Theory by Srivastava [9]. After that, several researches have been done by many authors, which has played an important role in the development of Geometric Function Theory. In particular, Srivastava and Bansal [10] studied the close-to-convexity of q-Mittag-Leffler functions. Srivastava et al. [11, 12] also studied the class of q-starlike functions in the conic region, while the upper bound of third Hankel determinant for the class of q-starlike functions was investigated by Mahmood et al. [13]. Recently, a number of articles have been published by Mahmood et al. and Srivastava et al. [14-17], in which they studied the classes of q-starlike functions related with Janowski functions from many different aspects. Wongsaijai and Sukantamala [18] generalised certain subclasses of starlike functions in a systematic way. Additionally, a survey-cum-exposition review article [19] by Srivastava will be potentially useful for researchers working on the applications of the q-calculus in Geometric Function Theory of Complex Analysis. Indeed, in this survey-cum-exposition article, Srivastava also gave the mathematical explanation and application of fractional q-calculus and fractional q-differential operators in geometric function theory.

Our present investigation is motivated by the well-established potential for the usage of the basic (or q-) calculus and the fractional basic (or q-) calculus in Geometric Function Theory as described in a recently-published survey-cum-exposition review article by Srivastava [19], Wongsaijai and Sukantamala [18], Khan et al. [20] and other related works cited above in this article. We shall consider three new subfamilies of multivalent (p-valent) q-starlike functions with respect to higher-order q-derivatives. Several properties and characteristics, e.g. inclusion results, sufficient conditions, distortion theorems and radius problem, shall be discussed in this investigation. We shall also point out some relevant connections of our results with the existing results.

Let $\mathcal{A}(p)$ denote the class of functions of the form

$$\omega(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathsf{N} = \{1,2,\ldots\}) , \tag{1}$$

which are analytic and p-valent in the open unit disk:

$$\mathsf{U} = \{z \ : \ z \in \mathsf{C} \quad \text{and} \quad |z| < 1\},$$

where C is a set of complex numbers. In particular, we write

$$\mathcal{A}(1) = \mathcal{A}.$$

Furthermore, by $\mathcal{S} \subset \mathcal{A}$ we shall denote the class of all functions which are univalent in U. The extension of any result from univalent to p-valent may be trivial or extremely difficult or perhaps false. There are several results that generalise classical results on univalent functions to the corresponding p-valent results. The first success in obtaining sharp inequalities for multivalent functions was attained by Hayman [21]. Several other researchers, for example Noor et. al. [22] and Noor and Khan [23], have also presented their findings on this subject.

The familiar class of p-valently starlike functions in U is denoted by $\mathbf{S}^*(p)$, which consists of functions $\omega \in \mathbf{A}(p)$ that satisfy the following condition:

$$\Re\left(\frac{z\omega'(z)}{\omega(z)}\right) > 0 \qquad (\forall z \in \mathsf{U}). \tag{2}$$

One can easily see that

$$\mathbf{S}^*(1) = \mathbf{S}^*,$$

where $\mathbf{S}^*$ is the well-known class of starlike functions.

SET OF DEFINITIONS

In this section we first give some basic definitions and concept details of q-calculus. Thereafter we demonstrate three (presumably new) subclasses of the class $\mathbf{S}_q^*$ of q-starlike functions with higher-order q-derivatives. Also, throughout the article unless otherwise mentioned, we suppose that $0 < q < 1$ and $p \in \mathsf{N} = \{1, 2, 3 \ldots\}$.

Definition 1. Let $q \in (0,1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \frac{1-q^\lambda}{1-q} & (\lambda \in \mathsf{C}) \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + \ldots + q^{n-1} & (\lambda = n \in \mathsf{N}). \end{cases}$$

Definition 2. Let $q \in (0,1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1 & (n = 0) \\ \prod\limits_{k=1}^{n} [k]_q & (n \in \mathsf{N}). \end{cases}$$

Definition 3 [1,2]**.** The q-derivative (or q-difference) D_q of a function ω in a given subset of C is defined by

$$(\mathsf{D}_q\omega)(z) = \begin{cases} \frac{\omega(z)-\omega(qz)}{(1-q)z} & (z \neq 0) \\ \omega'(0) & (z = 0) \end{cases} \tag{3}$$

provided that $\omega'(0)$ exists.

From Definition 3 we can observe that

$$\lim_{q \to 1-} (\mathsf{D}_q\omega)(z) = \lim_{q \to 1-} \frac{\omega(z) - \omega(qz)}{(1-q)z} = \omega'(z),$$

for a differentiable function ω in a given subset of C. It is readily known from (1) and (3) that

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$$\left(\mathbf{D}_q^{(1)}\omega\right)(z)=[p]_q z^{p-1}+\sum_{n=1}^{\infty}[n+p]_q a_{n+p}z^{n+p-1}$$

$$\left(\mathbf{D}_q^{(2)}\omega\right)(z)=[p]_q[p-1]_q z^{p-2}+\sum_{n=1}^{\infty}[n+p]_q[n+p-1]_q a_{n+p}z^{n+p-2}$$

$$\begin{matrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & & \cdot \end{matrix} \tag{4}$$

$$\left(\mathbf{D}_q^{(p)}\omega\right)(z)=[p]_q!+\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n]_q!}a_{n+p}z^{n},$$

where $\left(\mathbf{D}_q^{(p)}\omega\right)(z)$ is the p^{th} q- derivative of $\omega(z)$.

Definition 4 [8]. A function $\omega\in\mathbf{A}$ is said to belong to the class $\mathbf{S}_q^*$ if

$$\omega(0)=\omega'(0)-1=0 \tag{5}$$

and

$$\left|\frac{z}{\omega(z)}\left(\mathbf{D}_q\omega\right)z-\frac{1}{1-q}\right|\le\frac{1}{1-q}. \tag{6}$$

It is readily observed that as $q\to 1-$, the closed disk given by

$$\left|w-\frac{1}{1-q}\right|\le\frac{1}{1-q}$$

becomes the right-half plane and then the class $\mathbf{S}_q^*$ of q-starlike functions reduces to the familiar class $\mathbf{S}^*$.

We now introduce three (presumably new) subclasses of the class $\mathbf{S}_q^*$ of q-starlike functions with higher-order q-derivatives in the following way.

Definition 5. A function $\omega\in\mathbf{A}(p)$ is said to belong to the class $\mathbf{S}_q^*(1,\eta,p)$ if and only if

$$\Re\left(\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}\right)>\eta \qquad (0<\eta<1).$$

We call $\mathbf{S}_q^*(1,\eta,p)$ the class of multivalent (p-valent) q-starlike functions of Type 1.

Definition 6. A function $\omega\in\mathbf{A}(p)$ is said to belong to the class $\mathbf{S}_q^*(2,\eta,p)$ if and only if

$$\left|\frac{\dfrac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}-\eta}{1-\eta}-\frac{1}{1-q}\right|<\frac{1}{1-q} \qquad (0<\eta<1).$$

We call $\mathbf{S}_q^*(2,\eta,p)$ the class of multivalent (p-valent) q-starlike functions of Type 2.

Definition 7. A function $\omega\in\mathbf{A}(p)$ is said to belong to the class $\mathbf{S}_q^*(3,\eta,p)$ if and only if

$$\left|\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}-1\right|<1-\eta \qquad (0<\eta<1).$$

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We call $\mathsf{S}_q^*(3,\eta,p)$ the class of multivalent (p-valent) q-starlike functions of Type 3.

Each of the following special cases of the above defined q-starlike function classes

$$\mathsf{S}_q^*(1,\eta,p),\quad \mathsf{S}_q^*(2,\eta,p) \quad\text{and}\quad \mathsf{S}_q^*(3,\eta,p)$$

is worthy of note:

I. One can easily see that

$$\mathsf{S}_q^*(1,\eta,1)=\mathsf{S}_{(q,1)}^*(\eta),\qquad \mathsf{S}_q^*(2,\eta,1)=\mathsf{S}_{(q,2)}^*(\eta),\qquad \mathsf{S}_q^*(3,\eta,1)=\mathsf{S}_{(q,3)}^*(\eta),$$

where $\mathsf{S}_{(q,1)}^*(\eta)$, $\mathsf{S}_{(q,2)}^*(\eta)$ and $\mathsf{S}_{(q,3)}^*(\eta)$ are the function classes introduced and studied by Wongsaijai and Sukantamala [18]. It should also be noted that

$$\mathsf{S}_{(q,2)}^*(\eta)=\mathsf{S}_q^*(\eta),$$

where $\mathsf{S}_q^*(\eta)$ is the function class introduced and studied by Agrawal and Sahoo [24].

II. If we put

$$\eta=0 \quad\text{and}\quad p=1$$

in Definition 2, we get the class S_q^* introduced and studied by Ismail et al. [8].

Geometrically, for $\omega\in\mathsf{S}_q^*(m,\eta,p)$ $(m=1,2,3)$, the quotient $\dfrac{z\mathsf{D}_q^{(p)}\omega(z)}{\mathsf{D}_q^{(p-1)}\omega(z)}$ lies in the domains Ω_j $(j=1,2,3)$ given by

$$\Omega_1=\{\Upsilon\ :\ \Upsilon\in\mathsf{C}\ \text{ and }\ \Re(\Upsilon)>\eta\},$$

$$\Omega_2=\left\{\Upsilon\ :\ \Upsilon\in\mathsf{C}\ \text{ and }\ \left|\Upsilon-\frac{1-\eta q}{1-q}\right|<\frac{1-\eta}{1-q}\right\}$$

and

$$\Omega_3=\{\Upsilon\ :\ \Upsilon\in\mathsf{C}\ \text{ and }\ |\Upsilon-1|<1-\eta\}$$

respectively.

In this article many properties and characteristics, for example sufficient conditions, inclusion results, distortion theorems and radius problem, are discussed. Relevant connections of our results with a number of other related works on this subject have also been indicated.

MAIN RESULTS

We first derive the inclusion results for the following generalised q-starlike function classes:

$$\mathsf{S}_q^*(1,\eta,p),\quad \mathsf{S}_q^*(2,\eta,p) \quad\text{and}\quad \mathsf{S}_q^*(3,\eta,p),$$

which involve higher-order q-derivatives.

Theorem 1. *If* $0<\eta<1$, *then*

$$\mathsf{S}_q^*(3,\eta,p)\subset\mathsf{S}_q^*(2,\eta,p)\subset\mathsf{S}_q^*(1,\eta,p).$$

Proof. First we suppose that $\omega\in\mathsf{S}_q^*(3,\eta,p)$. Then by Definition 7 we have

$$\left|\frac{z\mathsf{D}_q^{(p)}\omega(z)}{\mathsf{D}_q^{(p-1)}\omega(z)}-1\right|<1-\eta.$$

Moreover, by using the triangle inequality we have

$$\left|\frac{\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}-\eta}{1-\eta}-\frac{1}{1-q}\right|=\frac{1}{1-\eta}\left|\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}-\eta-\frac{1-\eta}{1-q}\right|$$

$$\leq\frac{1}{1-\eta}\left|\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}-1\right|+\frac{q}{1-q}\leq 1+\frac{q}{1-q} \tag{7}$$

$$=\frac{1}{1-q}.$$

The inequalities in (7) show that $\omega\in\mathbf{S}_q^*(2,\eta,p)$, so $\mathbf{S}_q^*(3,\eta,p)\subset\mathbf{S}_q^*(2,\eta,p)$. Next we suppose $\omega\in\mathbf{S}_q^*(2,\eta,p)$. Then by Definition 6 we have

$$\left|\frac{\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}-\eta}{1-\eta}-\frac{1}{1-q}\right|<\frac{1}{1-q},$$

which can also be written as

$$\left|\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}-\frac{1-\eta q}{1-q}\right|<\frac{1-\eta}{1-q}. \tag{8}$$

From (8) we see that $\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}$ lies in the circle of radius $\frac{1-\eta}{1-q}$ centred at $\frac{1-\eta q}{1-\eta}$ and we observe that

$$\frac{1-\eta q}{1-q}-\frac{1-\eta}{1-q}=\eta,$$

which implies that

$$\Re\left(\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}\right)>\eta,$$

so $\omega\in\mathbf{S}_q^*(1,\eta,p)$; that is $\mathbf{S}_q^*(2,\eta,p)\subset\mathbf{S}_q^*(1,\eta,p)$. This completes the proof of Theorem 1.

As a special case of Theorem 1, if we put $p=1$, we get the following known result according to Wongsaijai and Sukantamala [18].

Corollary 1 [18]**.** *For* $0<\eta<1$,

$$\mathbf{S}_{q,3}^*(\eta)\subset\mathbf{S}_{q,2}^*(\eta)\subset\mathbf{S}_{q,1}^*(\eta).$$

Finally, by means of a coefficient inequality, we give a sufficient condition for the class $\mathbf{S}_q^*(3,\eta,p)$ of generalised q-starlike functions of Type 3, which also provides a corresponding sufficient condition for the classes $\mathbf{S}_q^*(1,\eta,p)$ and $\mathbf{S}_q^*(2,\eta,p)$ of Type 1 and Type 2 respectively.

Theorem 2. *A function* $\omega\in\mathbf{A}(p)$ *and of the form* (1) *is in the class* $\mathbf{S}_q^*(3,\eta,p)$ *if it satisfies the following coefficient inequality*:

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$$\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n]_q!}\left(\frac{[n+1]_q-\eta}{[n+1]_q}\right)|a_{n+p}|<(1-\eta)[p]_q!. \tag{9}$$

Proof. Assuming that (9) holds true, it suffices to show that

$$\left|\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}-1\right|<1-\eta.$$

We have

$$\begin{aligned}\left|\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}-1\right| &= \left|\frac{z\mathbf{D}_q^{(p)}\omega(z)-\mathbf{D}_q^{(p-1)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}\right| \\ &= \left|\frac{\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n]_q!}\left(1-\frac{1}{[n+1]_q}\right)a_{n+p}z^{n+1}}{[p]_q!z+\sum_{n=1}^{\infty}\frac{[n+p]_q}{[n+1]_q}a_{n+p}z^{n+1}}\right| \\ &\le \frac{\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n]_q!}\left(1-\frac{1}{[n+1]_q}\right)|a_{n+p}||z^{n+1}|}{[p]_q!|z|-\sum_{n=1}^{\infty}\frac{[n+p]_q}{[n+1]_q}|a_{n+p}||z^{n+1}|} \le \frac{\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n]_q!}\left(1-\frac{1}{[n+1]_q}\right)|a_{n+p}|}{[p]_q!-\sum_{n=1}^{\infty}\frac{[n+p]_q}{[n+1]_q}|a_{n+p}|}.\end{aligned} \tag{10}$$

The last expression in (10) is bounded above by $1-\eta$ if

$$\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n]_q!}\left(\frac{[n+1]_q-\eta}{[n+1]_q}\right)|a_{n+p}|<(1-\eta)[p]_q!,$$

which completes the proof of Theorem 2.

ANALYTIC FUNCTIONS WITH NEGATIVE COFFICENTS

In this section we introduce new subclasses of multivalent q-starlike functions which involve negative coefficients. Let $\mathbf{T}$ be a subset of $\mathbf{A}(p)$ consisting of functions with negative coefficients, i.e.

$$\omega(z)=z^p-\sum_{n=1}^{\infty}|a_{n+p}|z^{n+p}. \tag{11}$$

We also let

$$\mathbf{TS}_q^*(m,\eta,p) := \mathbf{S}_q^*(m,\eta,p)\cap\mathbf{T} \quad (m=1,2,3). \tag{12}$$

Theorem 3. *If* $0\le\eta<1$, *then*

$$\mathbf{TS}_q^*(1,\eta,p)\equiv\mathbf{TS}_q^*(2,\eta,p)\equiv\mathbf{TS}_q^*(3,\eta,p).$$

Proof. In view of Theorem 1, it is sufficient here to show that

$$\mathbf{TS}_q^*(1,\eta,p)\subset\mathbf{TS}_q^*(3,\eta,p).$$

Indeed, if we assume that $\omega\in\mathbf{TS}_q^*(1,\eta,p)$, then we have

$$\Re\left(\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}\right)\le\eta.$$

We now consider

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$$\Re\left(\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}\right)=\Re\left(\frac{[p]_q!z-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n]_q!}\left|a_{n+p}\right|z^{n+1}}{[p]_q!z-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n+1]_q!}\left|a_{n+p}\right|z^{n+1}}\right)$$
$$\Re\left(\frac{[p]_q!-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n]_q!}\left|a_{n+p}\right|z^{n}}{[p]_q!-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n+1]_q!}\left|a_{n+p}\right|z^{n}}\right)>\eta. \tag{13}$$

If we suppose that z lies on the real axis, then the value of $\frac{z\mathbf{D}_q^{(p)}\omega(z)}{\mathbf{D}_q^{(p-1)}\omega(z)}$ is real. In this case, by letting $z \to 1-$ on the real line, we have

$$[p]_q!-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n]_q!}\left|a_{n+p}\right|>\eta\left([p]_q!-\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n+1]_q!}\left|a_{n+p}\right|\right). \tag{14}$$

We see that (14) satisfies the inequality in (9). So by applying Theorem 2, the proof of Theorem 3 is completed.

As a special case of Theorem 3, if we put $p=1$, then we have the following known result.

Corollary 2 [18]. *If* $0 \le \eta < 1$, *then*

$$\mathbf{TS}^{*}_{(q,1)}(\eta)\equiv\mathbf{TS}^{*}_{(q,2)}(\eta)\equiv\mathbf{TS}^{*}_{(q,3)}(\eta).$$

The assertions of Theorem 3 imply that Type 1, Type 2 and Type 3 of the generalised *p*-valently *q*-starlike functions are exactly the same. For convenience, therefore, we state the following distortion theorem by using the notation $\mathbf{TS}^{*}_{q}(m,\eta,p)$, in which it is tacitly assumed that $m=1,2,3$.

Theorem 4. *If* $\omega\in\mathbf{TS}^{*}_{q}(m,\eta,p)$ $(m=1,2,3)$, *then*

$$r^p-\left(\frac{(1-\eta)[p]_q!}{\frac{[1+p]_q!}{[1]_q!}\left(\frac{[2]_q-\eta}{[2]_q}\right)}\right)r^{p+1}\le|\omega(z)|\le r^p+\left(\frac{(1-\eta)[p]_q!}{\frac{[1+p]_q!}{[1]_q!}\left(\frac{[2]_q-\eta}{[2]_q}\right)}\right)r^{p+1}\qquad(n\in\mathsf{N})$$
$$\left(\left|z^p\right|=r^p\quad(0<r<1)\right).$$

Proof. We note that the following inequality follows from Theorem 2:

$$\frac{[1+p]_q!}{[1]_q!}\left(\frac{[2]_q-\eta}{[2]_q}\right)\sum_{n=1}^{\infty}\left|a_{n+p}\right|\le\sum_{n=1}^{\infty}\frac{[n+p]_q!}{[n]_q!}\left(\frac{[n+1]_q-\eta}{[n+1]_q}\right)\left|a_{n+p}\right|$$
$$<(1-\eta)[p]_q!$$

so that

$$|\omega(z)|\le r^p+\sum_{n=1}^{\infty}\left|a_{n+p}\right|r^{n+p}\le r^p+r^{p+1}\sum_{n=1}^{\infty}\left|a_{n+p}\right|\le r^p+\frac{(1-\eta)[p]_q!}{\frac{[1+p]_q!}{[1]_q!}\left(\frac{[2]_q-\eta}{[2]_q}\right)}r^{p+1}.$$

Similarly, we have

$$|\omega(z)|\ge r^p-\sum_{n=1}^{\infty}\left|a_{n+p}\right|r^{n+p}\ge r^p-r^{p+1}\sum_{n=1}^{\infty}\left|a_{n+p}\right|\ge r^p-\frac{(1-\eta)[p]_q!}{\frac{[1+p]_q!}{[1]_q!}\left(\frac{[2]_q-\eta}{[2]_q}\right)}r^{p+1}.$$

We have thus completed the proof of Theorem 4.

In its special case when $p=1$ and if we let $q \to 1-$, Theorem 4 reduces to the following known result.

Corollary 3 [25]. *If* $\omega \in \mathtt{TS}^*(\eta)$, *then*

$$r - \left(\frac{1-\eta}{2-\eta}\right) r^2 \le |\omega(z)| \le r + \left(\frac{1-\eta}{2-\eta}\right) r^2 \quad (|z| = r \ \ (0<r<1)).$$

The following result (Theorem 5) can be proved by using arguments similar to those already presented in the proof of Theorem 4, so we choose to omit the details of our proof of Theorem 5.

Theorem 5. *If* $\omega \in \mathtt{TS}_q^*(m,\eta,p) \ \ (m=1,2,3)$, *then*

$$pr^{p-1} - \left(\frac{(p+1)(1-\eta)[p]_q!}{\frac{[1+p]_q!}{[1]_q!}\left(\frac{[2]_q-\eta}{[2]_q}\right)}\right) r^p \le |\omega'(z)| \le pr^{p-1} + \left(\frac{(p+1)(1-\eta)[p]_q!}{\frac{[1+p]_q!}{[1]_q!}\left(\frac{[2]_q-\eta}{[2]_q}\right)}\right) r^p$$

$$(|z|^p = r^p \ \ (0<r<1)).$$

In its special case if we put $p=1$ and let $q \to 1-$, Theorem 5 reduces to the following known result.

Corollary 4 [25]. *If* $\omega \in \mathtt{TS}^*(\eta)$, *then*

$$1 - \left(\frac{2(1-\eta)}{2-\eta}\right) r \le |\omega'(z)| \le 1 + \left(\frac{2(1-\eta)}{2-\eta}\right) r \quad (|z| = r \ \ (0<r<1)).$$

Finally, we determine the radii of p-valent starlikeness for functions in the class $\mathtt{TS}_q^*(m,\eta,p) \ \ (m=1,2,3)$.

Theorem 6. *Let function* ω *given by* (11) *be in the class* $\mathtt{TS}_q^*(m,\eta,p) \ \ (m=1,2,3)$. *If*

$$\inf_{n\ge 1}\left[\frac{p![n+p]_q!([n+1]_q-\eta)}{(n+p)![n]_q![n+1]_q(1-\eta)[p]_q!}\right]^{\frac{1}{n}} = r_0 \tag{15}$$

is positive, then function ω *is* p *-valently starlike in* $|z| \le r_0$.

Proof. To prove Theorem 6, it is sufficient to show that

$$\left|\frac{\omega^{(p)}(z)}{p!} - 1\right| < 1 \quad (|z| \le r_0).$$

Now, we have

$$\left|\frac{\omega^{(p)}(z)}{p!} - 1\right| = \left|-\sum_{n=1}^{\infty}\frac{(n+p)!}{p!}|a_{n+p}|z^n\right|$$

$$\le \sum_{n=1}^{\infty}\frac{(n+p)!}{p!}|a_{n+p}||z|^n.$$

Thus

$$\left|\frac{\omega^{(p)}(z)}{p!} - 1\right| < 1$$

if

$$\sum_{n=1}^{\infty} \frac{(n+p)!}{p!} |a_{n+p}||z|^n < 1. \tag{16}$$

In light of Theorem 2, the inequality in (16) will be true if

$$\frac{(n+p)!}{p!} |z|^n \leq \frac{[n+p]_q!([n+1]_q - \eta)}{[n]_q![n+1]_q(1-\eta)[p]_q!}. \tag{17}$$

Solving the inequality in (17) for z, we have

$$|z| \leq \left(\frac{p![n+p]_q!([n+1]_q - \eta)}{(n+p)![n]_q![n+1]_q(1-\eta)[p]_q!} \right)^{\frac{1}{n}}. \tag{18}$$

In view of (18), the proof of Theorem 6 is complete.

Theorem 7. *Let function* ω *given by* (11) *be in the class* $\mathbf{TS}_q^*(m,\eta,p)$ $(m=1,2,3)$. *If*

$$\inf_{n\geq 1} \left[\frac{(1-\eta)[p]_q[n]_q![n+1]_q\, p}{[n+p]_q!([n+1]_q - \eta)(n+p)} \right]^{\frac{1}{n}} = r_0$$

is positive, then function ω *is* p *-valently close-to-convex in* $|z| \leq r_0$.

Proof. By applying Theorem 2 and the form (11), we see that, for $|z| < r_0$, we have

$$\left| \frac{\omega'(z)}{z^{p-1}} - p \right| < p \qquad (|z| \leq r_0),$$

which completes the proof of Theorem 7.

Corollary 5. *Let function* ω *given by* (11) *be in the class* $\mathbf{TS}_q^*(m,\eta,p)$ $(m=1,2,3)$. *If*

$$\inf_{n\geq 1} \left[\frac{p![n+p]_q!([n+1]_q - \eta)}{(n+p)![n]_q![n+1]_q(1-\eta)(n+p)[p]_q!} \right]^{\frac{1}{n}} = r_0$$

is positive, then function ω *is* p *-valently convex in* $|z| \leq r_0$.

CONCLUSIONS

We have first defined certain new subclasses of q-starlike functions and then derived their many properties and characteristics. For validity and verification of our results, relevant connections with those in earlier works have been pointed out.

Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas.

Moreover, in this recently-published survey-cum-exposition review by Srivastava [19], the so-called (p,q)-calculus has been exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant. This observation will indeed apply also to any attempt to produce the rather straightforward (p,q)-variations of the results which we have presented in this paper.

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