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Full Paper

Time-frequency plane behavioural studies of harmonic and chirp functions with fractional Fourier transform (FRFT)

Renu Jain[1], Rajiv Saxena[2] and Rajshree Mishra[3,*]

[1] School of Mathematics and Allied Sciences (SOMAAS), Jiwaji University, Gwalior (M.P.) India
[2] Jaypee Institute of Engineering and Technology (JIET), Guna (M.P.) India
[3] School of Mathematics and Allied Sciences (SOMAAS), Jiwaji University, Gwalior (M.P.) India
[*] Corresponding author, e-mail: rajshreemishraa@gmail.com

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Abstract: The behaviour of harmonic and chirp functions was studied on the time-frequency plane with the help of fractional Fourier transform (FRFT). Studies were also carried out through simulation with different numbers of samples of the functions. Variations were observed in the maximum side-lobe level (MSLL), half main-lobe width (HMLW) and side-lobe fall-off rate (SLFOR) of the functions. The parameters of these functions were compared with a similar set of parameters of some of the popular window functions. It can thus be concluded that in the time-frequency plane, the chirp function provides better spectral parameters than those of Boxcar window function with some particular values of rotational angle. A similar type of inference can also be drawn for the harmonic function in the time-frequency plane. Of course the rotational angle might vary in this case and a comparative analysis was carried out with Fejer window and the cosine-tip window functions. This study may prove to be helpful in replacing these existing window functions in a variety of applications where a particular parameter or group of parameters of the harmonic and chirp functions are found superior.

Keywords: harmonic function, chirp function, fractional Fourier transform (FRFT), window function, spectral parameters, time-frequency plane

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Introduction

The Fourier transform (FT) is the most frequently and extensively used tool in signal processing and analysis [1]. Fractional Fourier transform, generally known as FRFT, is a generalisation of FT with an order parameter `*a*'. Mathematically, the a^{th} order FRFT F^a is the a^{th} power of ordinary FT operation *F*. The first order *(a=1)* FRFT is the ordinary FT and the zeroth order FRFT is the identity transformation.

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FRFT has now become the most frequently and extensively used tool in the area of signal processing, analysis and optics. It also finds many applications in differential equations, optical beam propagation, quantum mechanics, statistical optics and optical signal processing [2-4]. In fact, in every area in which FT and frequency domain concepts are utilised, there exists a provision for the generalisation and improvisation by applying FRFT.

Fractionalisation of the transform started in 1929 by Wiener [5] by solving of the differential equations. Condon in 1937 [6] and Bargmann in 1961 [7] followed his work. Namias in 1980 and Mcbride and Kerr in 1987 did remarkable work in developing the transform, its algebra and calculus [8-9].

The definition of continuous-time FRFT given by Namias is validated to date. In 1993 Almeidia came out with the time-frequency representation of FRFT [10]. Almeidia, Zayed and Mustard contributed in developing the properties of FRFT [11-13]. On the other hand, Ozaktas and his team and many other researchers worked on computation of FRFT and on discrete version of FRFT [14-15]. Due to diversified applications of FRFT many researchers are continuously contributing in developing the transform, its discrete version and applications [3, 4, 16]. During the last 15 years an enormous amount of research publications have been noticed in this area and FRFT is now established as a very powerful tool for almost all scientific and engineering applications.

In spectral analysis of a function, three parameters are of prime importance, i.e. the maximum side-lobe level (MSLL), the half main-lobe width (HMLW) and the side-lobe fall-off rate (SLFOR). An improvement in these parameters can improve the spectral performance of the function. Figure 1 provides a graphical representation of these parameters.

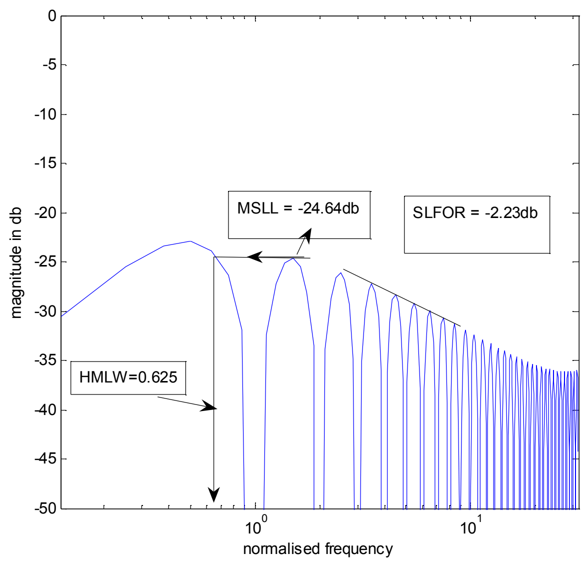

Figure 1. Parameters of harmonic function

The main objective of this paper is to study and analyse the behavioural changes of harmonic and chirp functions in the fractional Fourier domain. The harmonic function is defined by *exp* $(2\pi\, ix)$ and the chirp function is defined by *exp* $(-i\pi/4)$ *exp* $(i\pi x^2)$ where the domain of the functions is a set of real numbers. We observed changes in the above three parameters of both functions in the FRFT domain, i.e. in time-frequency plane.

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Defining the Fractional Fourier Transform (FRFT)

The FRFT F^a of order $a \in R$ is a linear integral operator that maps a given function $f(x),\ x \in R$ on to $f_a(\xi)$.

$\xi \in R$ by

$$f_a(\xi) = F^a(\xi) = \int_{-\infty}^{\infty} K_a(\xi,\ x)\, f(x)\, dx \qquad \text{........ (1)}$$

where $K_a(\xi,\ x)$ is kernel of transform and defined as follows:

$$K_a(\xi,\ x) = C_\alpha\, exp\{-i\pi(\frac{2x\xi}{sin\alpha} - (x^2+\xi^2)cot\alpha)\} \qquad \text{........ (2)}$$

with

$$C_\alpha = \sqrt{1 - i\,cot\alpha} = \frac{exp(-i(\pi\, sgn(sin\alpha)/4 - \alpha/2))}{\sqrt{sin\alpha}} \qquad \text{........ (3)}$$

where $\alpha = a\frac{\pi}{2}$.

Equation (2) is defined only for a $\neq 2$ m, i.e. α is not a multiple of π.

$$K_a(\xi,\ x) = \delta(x-\xi)\ \ for\ a = 4m\ or\ \alpha = 2\,m\pi \qquad \text{........ (4)}$$

and $K_a(\xi,\ x) = \delta(x+\xi)\ \ for\ a = 4m \pm 2\ or\ \ \alpha = (m \pm 1)\pi$ .... (5)

where m is an integer.

Since $\alpha = a\frac{\pi}{2}$ appears in equations only in the argument of trigonometric functions then the definition is periodic in a (or α). Thus we will often limit our attention to the interval $a \in (-2, 2]\ or\ \alpha \in (-\pi, \pi]$ and sometimes $a \in [0, 4)\ or\ \alpha \in [0, 2\pi)$. When a is outside the interval $0 \leq |a| \leq 2$ we need simply replace a by it's modulo 4 equivalent.

Different cases can be tabulated, as given in Table 1.

Table 1. Some important cases of FRFT [16]

Operation on signal	**Value of parameter *a***	**Value of** $\alpha = a\pi/2$	**Kernel**	**Fractional operator**
FT operator	$4m + 1$	$(4m+1)\pi/2$	$exp(-i2\pi x\xi)$	$F^1 = F$
Parity operator	$4m + 2$	$(2m+1)\pi$	$\delta(\xi + x)$	$F^2 = P$
Identity operator	$4m$	$2m\pi$	$\delta(\xi - x)$	$F^{4m} = I$
Inverse Fourier Transform (IFT) operator	$4m + 3$	$(4m+3)\pi/2$	$exp(+i2\pi x\xi)$	$F^3 = F^{-1}$

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The discrete version of FRFT is known as discrete fractional Fourier transform (DFRFT) and evaluation of FRFT is done by using DFRFT computational techniques [15]. The DFRFT of a signal f(x) can be computed by following four steps:

(i) Multiply the function by a chirp (a function whose frequency linearly increases with time),
(ii) Take its Fourier transform with its argument scaled by cosec α ,
(iii) Again multiply with a chirp, and
(iv) Multiply with a complex constant.

FRFT is now a widely studied transform and it is observed that FRFT of a signal exists under the same conditions as those for FT. The analytical expressions for FRFT of some common signals have been calculated as presented in Table 2.

Table 2. FRFT of standard functions [4]

Function	**FRFT**
1	$\frac{e^{i\alpha/2}}{\sqrt{cos\alpha}}\, exp(-i\pi u^2 tan\alpha)$
$\delta(u-\xi)$	$\frac{e^{i(\alpha/2-\pi/4)}}{\sqrt{sin\alpha}}\, e^{i\pi(cot\alpha(u^2+\xi^2)-2u\xi cosec\alpha)}$
$\delta(u)$	$\frac{e^{-i(\alpha/2-\pi/4)}}{\sqrt{sin\alpha}}\, e^{i\pi u^2 cot\alpha}$
$exp(i2\pi\xi u)$	$\frac{e^{i\alpha/2}}{\sqrt{cos\alpha}}\, e^{-i\pi(tan\alpha(\xi^2+u^2)-2u\xi sec\alpha)}$
$exp(-\pi u^2)$	$exp(-\pi u^2)$
$exp(i\pi\chi u^2)$	$\frac{\sqrt{1+i\,tan\,\alpha}}{\sqrt{1+\chi\,cot\,\alpha}}\, e^{i\pi u^2[(\chi-tan\,\alpha)/(1+\chi\,tan\,\alpha)]}$

FRFT Analysis of Harmonic Function

Spectral parameters of the harmonic function were observed in fractional Fourier domain through simulation. In one of the studies, the fractional angle α was varied from 0 to π/2 keeping the length of the harmonic function constant. These simulation studies were made by using DFRFT computation techniques.

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Figure 2 shows harmonic function in time domain and Figure 3 shows how the harmonic function evolves into its Fourier domain as angle *α* increases from 0 to π/2. Figure 4 represents the simulation results of harmonic function in time-frequency plane, obtained by varying either the length of function (N) or by changing the value of fractional angle (*α*). Some of the simulation results are given in Table 3.

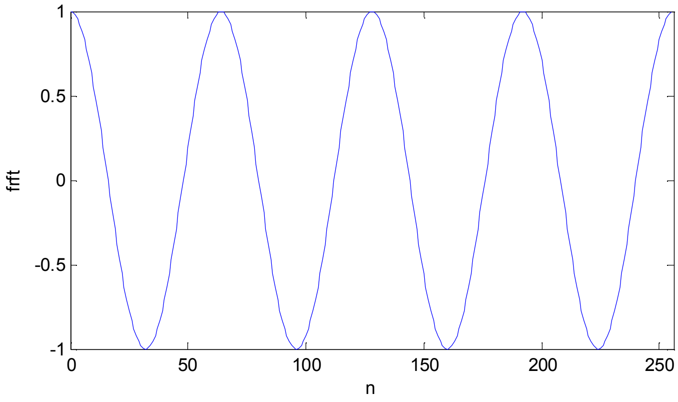

Figure 2. FRFT of harmonic function at *a*=0

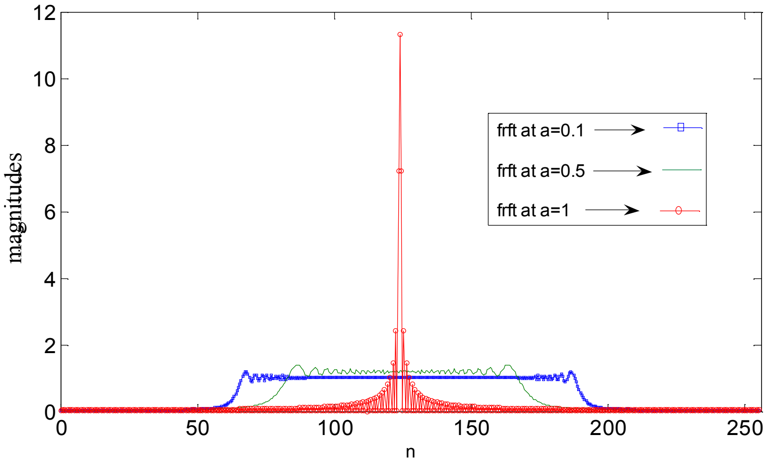

Figure 3. Magnitudes of FRFT of harmonic function by varying '*a*'

It was observed that for small sizes of signal (16, 32, 64), as fractional angle α increases in the time-frequency plane, MSLL of the function decreases in a regular manner, while a slight increment is noted in SLFOR. The HMLW of the function is also regular in nature except at a few particular points, and shows increments in the time-frequency plane (Figure 5-7). For large numbers of samples, the harmonic function shows irregular behaviour in general in the time-frequency plane. In this case, it shows regularity in its behaviour only in a smaller part of *α* domain.

During the studies of all the above cases of harmonic function in the time-frequency plane, some specific values of fractional angle α have been noted, whereat parameters of the harmonic function in FRFT domain are better than those of Fejer window function [17] in the sense of spectral performance.

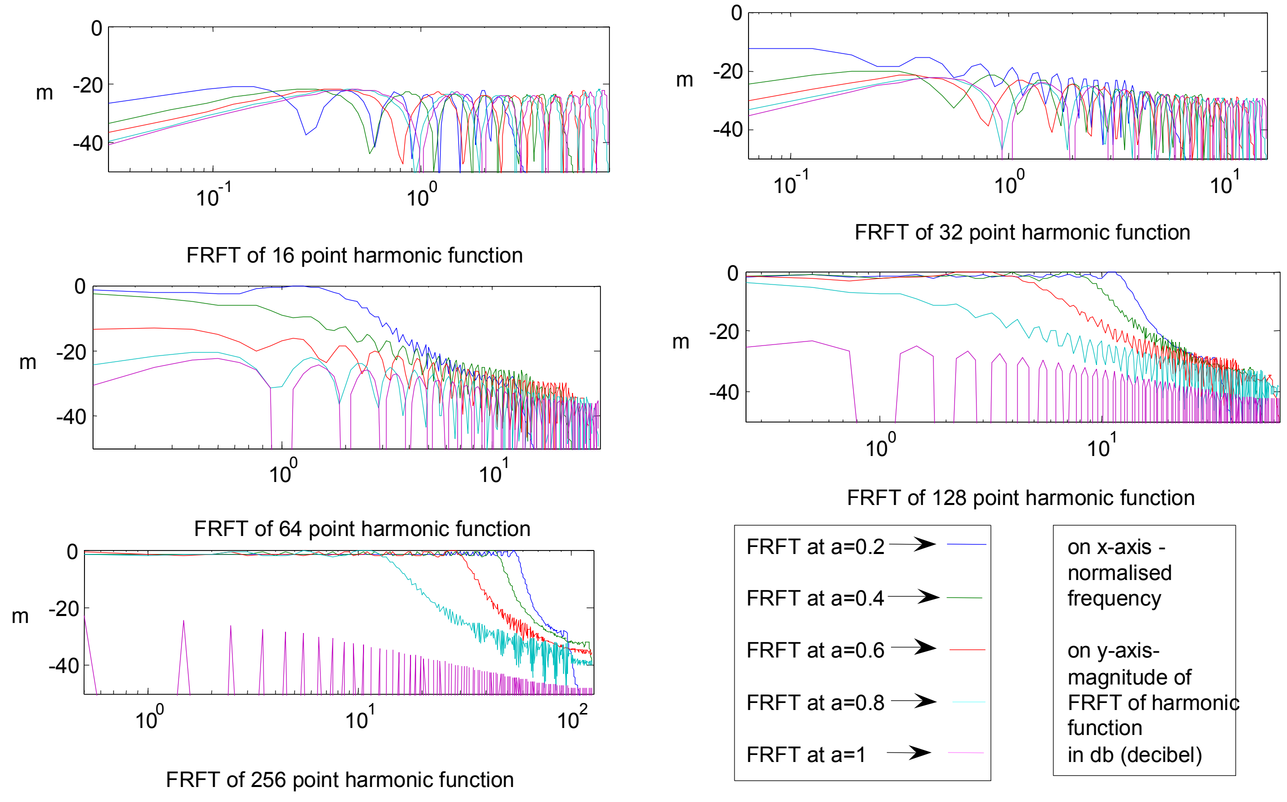

Figure 4. FRFT of harmonic function for different sizes of samples

Table 3. Parameters of FRFT of harmonic function

Fractional angle 'α'	**Sample under consideration**	**MSLL in db***	**HMLW**	**SLFOR in db***
3π/20	16	-22.68	0.1	-0.30
23π/50	16	-22.98	0.6563	-0.33
π/4	32	-23.16	0.45	-2.04
18π/50	32	-24	0.5625	-1.81
13π/50	64	-12	0.5	-3.42
7π/20	64	-19.2	0.625	-2.58
81π/200	128	-11.7	1.3	-4.5
9π/20	128	-17.32	0.75	-3.03
22π/50	256	-16.5	0.9	-4.2

* = decibel

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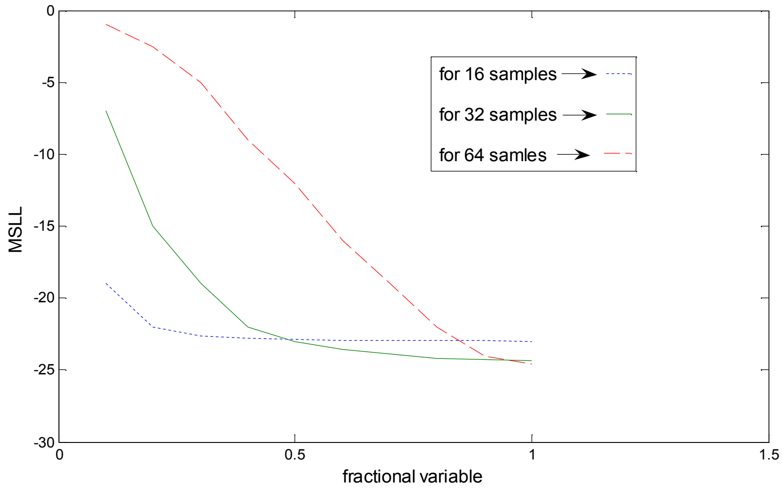

Figure 5. MSLL of FRFT of harmonic function for different sizes of samples

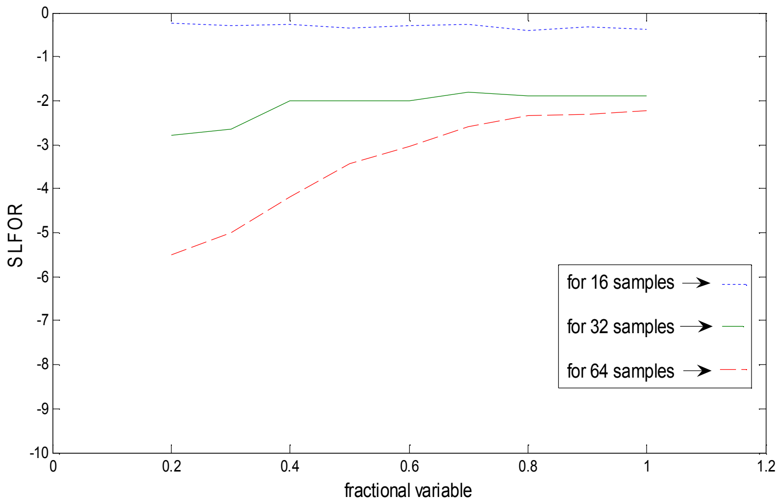

Figure 6. SLFOR of FRFT of harmonic function for different sizes of samples

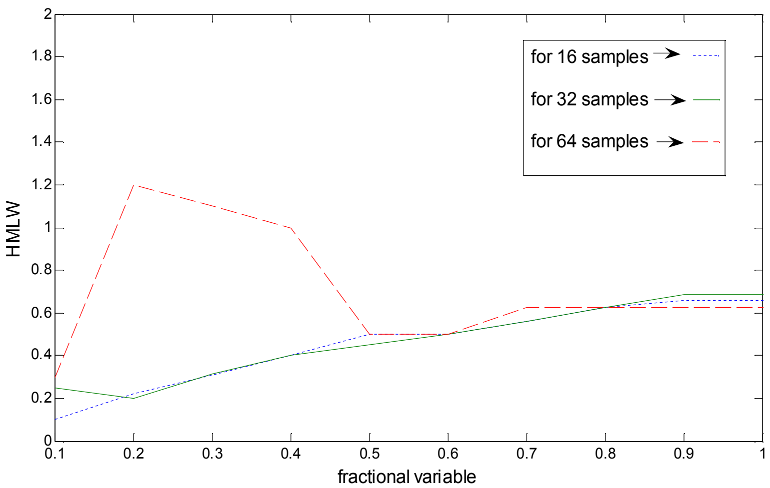

Figure 7. HMLW of FRFT of harmonic function for different sizes of samples

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FRFT Analysis of Chirp Function

FRFT of the chirp function was obtained and studied through simulation. Figure 8 shows chirp function in time domain while Figure 9 shows how the function transforms from time domain to frequency domain. Figure 10 shows the simulation results. Table 4 represents some of the simulation results for FRFT of a chirp function.

It was observed that in general the chirp function behaves in an irregular manner in time-frequency plane. Figure 10 indicates its nature in α domain for different sizes of samples and Figure 11 shows changes in MSLL of the function in fractional domain. Variation in MSLL increases in α domain when the number of samples is increased. In this case regular increments were noted only in a little part of α domain close to the angle $\alpha = \pi/2$.

HMLW of the function in general increases regularly with slight irregular variation for small samples, as shown in Figure12, while SLFOR of the same function also shows variation in an irregular manner but varies between -3 to -6 db. Large numbers of samples produced two specific results as discussed later.

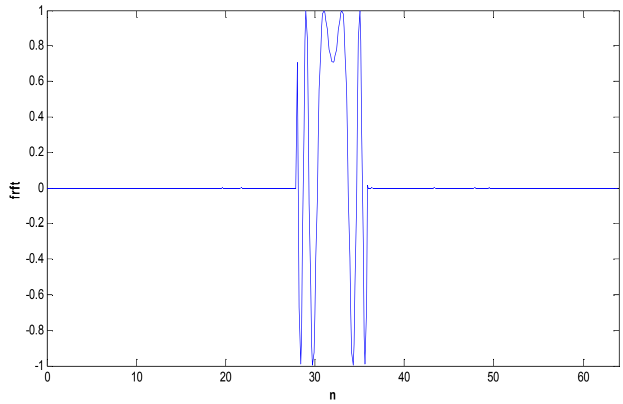

Figure 8. FRFT of chirp function at *a*=0

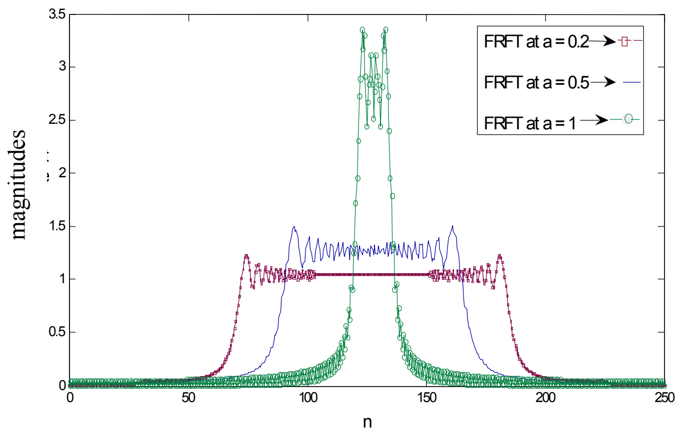

Figure 9. Magnitudes of FRFT of chirp function by varying '*a*'

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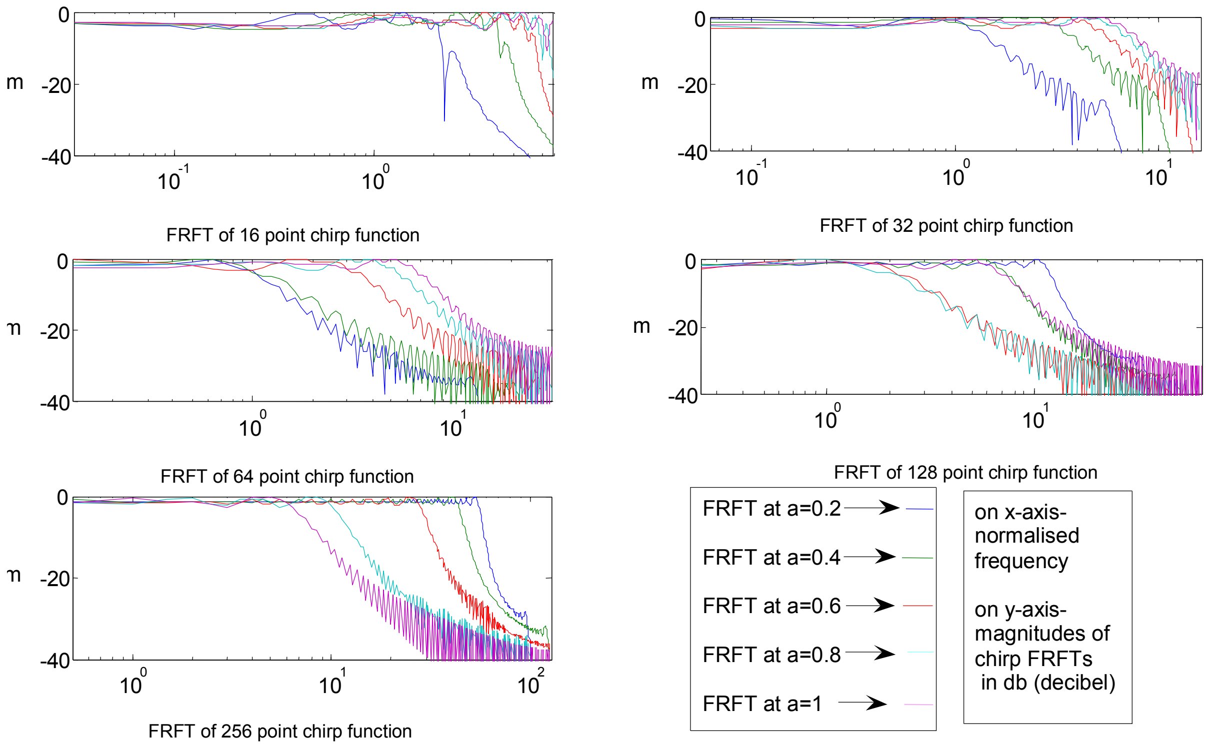

Figure 10. FRFT of chirp function for different sizes of samples

Table 4. Parameters of FRFT of chirp function

Fractional angle ‘α’	Samples under consideration	MSLL (in db*)	HMLW	SLFOR (in db*)
π/20	16	-9.63	0.8	-
π/4	16	-7.2	5.1	-
3π/20	32	-4.6	2.3	-5
7π/20	32	-3.4	5.3	-5.5
13π/50	64	-3.408	2.1	-6.8
3π/8	64	-5.3	4.50	-5.2
13π/50	128	-10	5.5	-4
23π/50	128	-13.09	6.75	-5.7

* = decibel

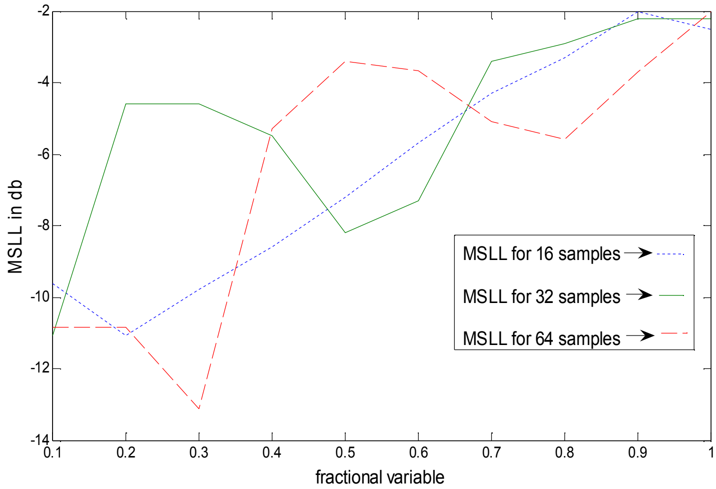

Figure 11. MSLL of FRFTs of chirp function for different sizes of samples

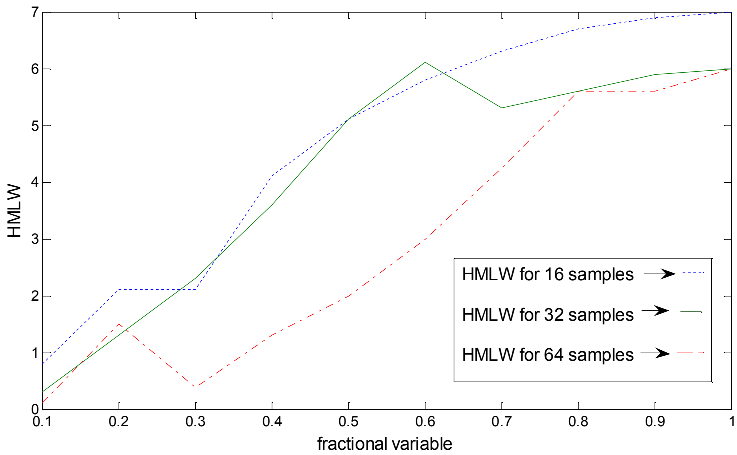

Figure 12. HMLW of FRFTs of chirp function for different sizes of samples

Discussion

It can be concluded that in the case of harmonic function, changes in HMLW, SLFOR and MSLL are in general regular in nature for small samples, but irregularity is observed when the size of samples is increased in the FRFT domain.

Two specific results for harmonic function for $\alpha = 41\pi/100$ and $9\pi/20$ were obtained during simulation studies and analysis, as shown in Figures 13-14. Parameters of these results were compared with those of cosine tip window function and Fejer window function. Superiority in results can be observed from Table 5. Frequency domain representation of harmonic function is included with its shifted version for comparison. These specific results can replace the window functions to generate a variety of applications and can also be applied according to the requirement of the spectral parameters.

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The behaviour of chirp function changes with the variation of fractional angle α with respect to the number of samples in time-frequency plane. This simulation studies have generated two important results with $\alpha = 3\pi/20$ and $\alpha = 71\pi/200$ for chirp function in fractional Fourier domain. Like the harmonic function, chirp function is not a window function but parameters of these two results are far better than the frequency domain parameters of Boxcar window function. Figure 15 shows db (decibel) plot of chirp function at $\alpha = 3\pi/20$ in time-frequency plane. Figure 16 shows db plot of the same function at $\alpha = 71\pi/200$. Comparative parameters are shown in Table 5. These two plots prove to be a better replacement for Boxcar window function and can be used as a tool for specific applications.

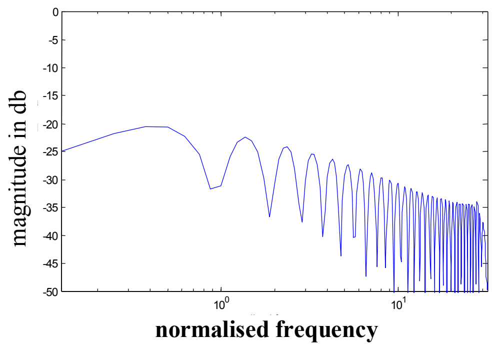

Figure 13. Magnitude of FRFT of harmonic function at a =0.82

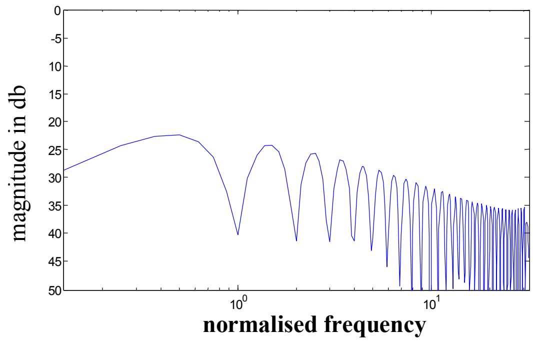

Figure 14. Magnitude of FRFT of harmonic function at a =0.9

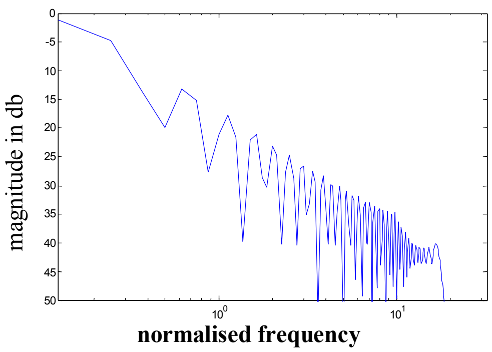

Figure 15. Magnitude of FRFT of chirp function at a =0.3

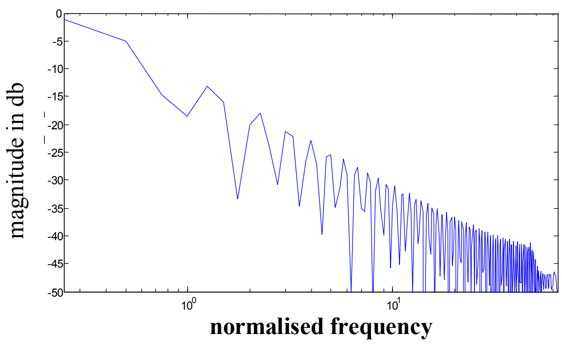

Figure 16. Magnitude of FRFT of chirp function at a =0.71

Table 5. Comparative parameters

Function	Fractional angle α	MSLL (in db)	HMLW	SLFOR (in db)
Harmonic	41 π/100	-23	0.625	-2.3
Cosine tip window	π/2	-23	1.35	-12
Harmonic	9 π/20	-24	0.625	-2.5
Fejer window	π/2	-26	1.63	-12
Chirp	3 π/20	-13.14	0.375	-5.23
Boxcar window	π/2	-13	0.81	-6
Chirp	71 π/200	-13.16	0.8	-5

Conclusions

The FRFT domain analysis of harmonic and chirp functions has been carried out and a few cases of these functions are observed to show better spectral performances in comparison to the existing weight functions. This clearly indicates that harmonic and chirp functions with some specific values of FRFT with order `*a*' provide a better candidate in applications wherever these weight functions are used.

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