Maejo Int. J. Sci. Technol. **2010**, *4*(01), 88-92

Maejo International
Journal of Science and Technology

ISSN 1905-7873
Available online at www.mijst.mju.ac.th

Full Paper

Application of separation of variables in Green's function to typical half-strip problem for elastic material

Paul J. Udoh* and Asuquo E. Eyo

Department of Mathematics, Statistics and Computer Science, University of Uyo, Nigeria

* Corresponding author, e-mail: udohpattie@yahoo.com

Received: 30 July 2009 / Accepted: 10 March 2010 / Published: 15 March 2010

Abstract: This paper presents the method of separation of variables in Green's function and its application to the problem of half-strip of elastic material. Unlike that of the full strip, the half-strip problem differs in specification of the boundary conditions. Though similar to the full-strip problem in conjugality conditions, the half-strip problem is amenable to the application of Poisson's equation instead of Laplace's equation. The results show that this problem is one-dimensional and the shearing stresses are non-symmetric.

Keywords: Green's function, half-strip pboblem, elastic material

---

Introduction

The method of separation of variables is the most general one for constructing Green's functions. This method was used to solve a full-strip elastic problem [1]. The approach here differs from that of Humphrey and Rajagopol [2] where the basic idea is that of decomposing the motion into two parts: one due to traction-free uniform heating and the other to isothermal mechanical loading.

However, in the method of separation of variables, the approach is holistic without analogous representation into singular and regular parts [3]. The method of separation of variables enables us to construct the Green's functions both for Laplace's and Poisson's operators [4] for simple boundary-value problems. This method allows us to represent the functions in an infinite series as is done in regularity conditions [5] and for the strain energy **W** [2]. The method adopted here enables us to construct and take into account the property of orthogonality for trigonometric functions in terms of one-dimensional differential equations.

Maejo Int. J. Sci. Technol. **2010**, *4*(01), 88-92

Mathematical Formulations

The Green's function G(x,ξ) for Poisson's equation $\nabla_x^2 \, G\,(x,\xi) = -\delta\,(x-\xi)$, and for the half-strip we specify the intervals as $0 < x_1 < \infty$ and $0 \le x_2 \le a_2$ under the following conditions:

$$\frac{\partial G}{\partial x_1} = 0 \;\; ; \; x_1 = 0, \;\; 0 \le x_2 \le \; a_2 \tag{1}$$

$$\frac{\partial G}{\partial x_1} = 0 \;\; ; \; x_2 = 0, \, a_2 \;\; ; \;\; 0 \le x_1 \le \; \infty \tag{2}$$

Here, the function G(x,ξ) should be bounded at infinity, i.e. $G\,|_{x_1 = 0} < \infty$

Computational Procedure

Following our earlier results [6], the solution to this problem is sought by means of the method of separation of variables leading to the following general trigonometric series:

$$G = a_0 + \sum_{n=1}^{\infty} a_n \cos v_1 x_2 + \sum_{m=1}^{\infty} b_m \sin v_1 x_2 \tag{3}$$

where, a_0, a_n and b_m are the functions of the independent variable x_i. The boundary conditions of this problem with respect to the independent variables x_2 reduce equation (3) with the general series to the following form:

$$G = a_0 + \sum_{n=1}^{\infty} a_n \cos v_1 x_2 \, , \;\; v_1 = \frac{n\pi}{a_2}, \; n = 1,2,3, \ldots\ldots\ldots\ldots \tag{4}$$

Substituting this expression in (4) for the Green's function G into the Poisson's $\nabla_x^2 \, G\,(x,\xi)$, we get

$$G = a_0'' + \sum_{n=1}^{\infty} (\; a_n'' - v_1^2 \, a_n) \cos v_1 x_2 \;\; = -\delta\,(x_1 - \xi_1)\,\delta\,(x_2 - \xi_2) \tag{5}$$

where, in the method of separation of variables, it takes the form:

$$\delta(x - \xi) = \delta(x_1 - \xi_1)\,\delta(x_2 - \xi_2) \tag{6}$$

Integrating equation (5) with respect to the independent variable x_2, we note that all the integrals with the exceptions of equations (7) and (8):

$$\int_0^{a_2} a_0'' \, d\,x_2 = a_0'' \, a_2 \tag{7}$$

$$\int_0^{a_2} \delta\,(x_1 - \xi_1)\,\delta\,(x_2 - \xi_2)\,dx_2 = \delta\,(x_1 - \xi_1) \tag{8}$$

are equal to zero. Therefore, to determine the function $a_0(x_1)$, we obtain the following differential equation of the second order:

$$a_0'' \;\; = -a_2^{-1}\,\delta\,(x_1 - \xi_1) \tag{9}$$

and the boundary condition and the condition at infinity:

$$a_0' \;(x_1 = 0)\,; \qquad a_0 \;(x_1 = \infty) < \;\infty \tag{10}$$

To construct the Green's function for this boundary-value problem, we use the standard technique [2]. The general solution of the 1D equation is sought in the form of

Maejo Int. J. Sci. Technol. **2010**, *4*(01), 88-92

$$a_2(x_1) = \begin{cases} c_1 x_1 + c_2, & x_1 \le \xi_1 \\ k_1 x_1 + k_2, & x_1 \ge \xi_1 \end{cases} \tag{11}$$

From the boundary condition and the condition at infinity, we obtain

$$a_0'(x_1 = 0) = 0 \Rightarrow c_1 = 0; \qquad a_0(x_1 = \infty) < \infty \Rightarrow k_1 = 0 \tag{12}$$

So the function $a_0(x_1)$ can be rewritten in the form:

$$a_2(x_1) = \begin{cases} c_2, & x_1 \le \xi_1 \\ k_2, & x_1 \ge \xi_1 \end{cases} \tag{13}$$

which satisfies the ordinary differential equation of the first order. Next, from the condition of conjugality at the point $x_1 = \xi_1$,

$$a_0(x_1 = \xi_1 - 0) = a_0(x_1 = \xi_1 - +0) \tag{14}$$

We obtain the following result: $c_2 = k_2 = b =$ constant.
Finally, for the desired function, we find

$$a_0(x_1, \xi_1) = \begin{cases} b, & x_1 \le \xi_1 \\ b, & x_1 \ge \xi_1 \end{cases} \tag{15}$$

Now, to determine the function a_m, we multiply both parts of the equation (5) by

$$\cos v_2 x_2 \quad (v_2 = s\pi / a_2, \ s = 1,2,3..............) \tag{16}$$

and take the integral with respect to the variable x_2.
Taking into account the property of orthogonality for trigonometric functions, we get the respective 1D differential equation:

$$(a_n'' - v_1^2 a_m)\frac{a_2}{2} = -\delta(x_1 - \xi_1)\cos v_1 \xi_2 \tag{17}$$

and the boundary conditions:

$$a_n'(x_1 = 0); \qquad a_n'(x_1 = \infty) < \infty \tag{18}$$

In solving equation (17), the following integrals have been calculated:

$$\int_0^{a_2} \cos v_2 x_2 \cos v_1 x_2 d x_2 = \begin{cases} 0, & v_1 \ne v_2 \\ a_0/2, & v_1 = v_2 \end{cases} \tag{19}$$

$$\int_0^{a_2} \delta(x_1 - \xi_1)\,\delta(x_2 - \xi_2)\cos v_2 x_2\, dx_2 = \delta(x_1 - \xi_1)\cos v_2 \xi_2 \tag{20}$$

By taking a notation, $a_m = 2a_2^{-1}\bar{a}_m \cos v_2 \xi_2$ (21)

with the conditions in (18), we reduce equation (19) to the boundary-value problem formulated to determine the function $\bar{a}_n$, i.e.

$$(a_n'' - v_1^2 a_m) = -\delta(x_1 - \xi_1) \tag{22}$$

$$a_n'(x_1 = 0) = 0 \tag{23}$$

$$a_n'(x_1 = \infty) < \infty \tag{24}$$

Maejo Int. J. Sci. Technol. **2010**, *4*(01), 88-92

Using the standard technique [7], the general solution of equation (11) is written in the form:

$$\bar{a}_n = \begin{cases} c_1 e^{-\nu_1 x_1} + c_2 e^{\nu_1 x_1} \;; & x_1 \le \xi_1 \\ k_1 e^{-\nu_1 x_1} + k_2 e^{\nu_1 x_1} \;; & x_1 \ge \xi_1 \end{cases} \tag{25}$$

Then from the conditions of conjugality at the point $x_1 = \xi_1$,

$$\bar{a}_n (x_1 = \xi_1 - 0) = \bar{a}_n (x_1 = \xi_1 + 0) \tag{26}$$

$$\bar{a}_n (x_1 = \xi_1 - 0) - \bar{a}_n (x_1 = \xi_1 + 0) = 1 \tag{27}$$

we get a system of two simultaneous linear algebraic equations:

$$(c_1 - k_1)\, e^{-\nu_1 \xi_1} + (c_2 - k_2)\, e^{\nu_1 \xi_1} = 0 \tag{28}$$

$$\nu_1 \left[(c_1 - k_1)\, e^{-\nu_1 \xi_1} - (c_2 - k_2)\, e^{\nu_1 \xi_1} \right] = -1 \tag{29}$$

Then from the boundary conditions and conditions at infinity, we get

$$\bar{a}_m (x_1 = 0) = 0 \Rightarrow c_2 - c_1 = 0 \tag{30}$$

$$\bar{a}'_n (x_1 = \infty) < \infty \Rightarrow k_2 = 0 \tag{31}$$

For the coefficients, we finally obtain the following values:

$$k_1 = \frac{e^{-\nu_1 \xi_1} - e^{\nu_1 \xi_1}}{2\nu_1}; \quad c_1 = c_2 = \frac{e^{-\nu_1 \xi_1}}{2\nu_1}; \quad k_2 = 0 \tag{32}$$

Using standard transformations, the Green's functions in equation (25) can be represented as

$$\bar{a}_n (x_1, \xi_1) = \begin{cases} \dfrac{1}{2\nu_1} \left(e^{\nu_1 (x_1 - \xi_1)} + e^{-\nu_1 (x_1 - \xi_1)} \right); & x_1 \le \xi_1 \\ \dfrac{1}{2\nu_1} \left(e^{-\nu_1 (x_1 - \xi_1)} + e^{-\nu_1 (x_1 - \xi_1)} \right); & x_1 \ge \xi_1 \end{cases} \tag{33}$$

On account of equation (21), the Green's function G(x, ξ) for the initial boundary-value problem can be written in the form:

$$G(x,\xi) = \begin{cases} G_\ell (x, \xi) = b + \dfrac{2}{a_2} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{2\nu_1} \left(e^{\nu_1 (x_1 - \xi_1)} + e^{-\nu_1 (x_1 - \xi_1)} \right) \cos \nu_1 x_2 \cos \nu_1 \xi_2; & x_1 \le \xi_1 \\ G_\ell (x, \xi) = b + \dfrac{2}{a_2} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{2\nu_1} \left(e^{\nu_1 (x_1 - \xi_1)} + e^{-\nu_1 (x_1 - \xi_1)} \right) \cos \nu_1 x_2 \cos \nu_1 \xi_2; & x_1 \ge \xi_1 \end{cases} \tag{34}$$

However, by making use of the known sum [1, 3, 4],

$$\sum_{n=1}^{\infty} \frac{p^n}{n} \cos n\alpha = -\ln \sqrt{1 - 2p\cos\alpha + p^2} \;;\; p^2 < 1,\; 0 \le \alpha \le 2\pi; \quad \text{or } p^2 \le 1,\; 0 \le \alpha \le 2\pi \tag{35}$$

and trigonometric formula:

$$\cos \nu_1 x_2 \cos \nu_1 \xi_2 = \frac{1}{2} \left[\cos \nu_1 (x_1 - \xi_2) + \cos \nu_1 (x_2 + \xi_2) \right], \tag{36}$$

then one can take the sum of the ordinary infinite series. After some computation, we obtain the final expression for the Green's function of the initial boundary-value Neumann's problem for the half-strip as:

$$G = b - \frac{1}{2\pi} \ln E E_1 E_2 E_{12} \tag{37}$$

where the functions E, E_1, E_2, and E_{12} are determined by the expressions:

Maejo Int. J. Sci. Technol. **2010**, *4*(01), 88-92

$$E = \sqrt{1 - 2e^{\frac{\pi}{a_2}(x_1 - \xi_1)} \cos\pi/a_2(x_2 - \xi_2) + e^{\frac{2\pi}{a_2}(x_1 - \xi_1)}} \quad (38)$$

$$E_1 = \sqrt{1 - 2e^{-\frac{\pi}{a_2}(x_1 + \xi_2)} \cos\pi/a_2(x_2 - \xi_2) + e^{\frac{-2\pi}{a_2}(x_1 + \xi_1)}} \quad (39)$$

$$E_2 = \sqrt{1 - 2e^{\frac{\pi}{a_2}(x_1 - \xi_1)} \cos\pi/a_2(x_2 + \xi_2) + e^{\frac{2\pi}{a_2}(x_1 - \xi_1)}} \quad (40)$$

$$E_{12} = \sqrt{1 - 2e^{-\frac{\pi}{a_2}(x_1 + \xi_2)} \cos\pi/a_2(x_2 + \xi_2) + e^{\frac{-2\pi}{a_2}(x_1 + \xi_1)}} \quad (41)$$

Conclusions

In this paper, we have made use of the method of separation of variables in Green's functions to solve the problem of the half-strip problem. A particular focus is given to the problem of the half-strip using Poisson's equation. For ease of formulation, we use trigonometric functions to construct the Green's functions for the Poisson's equation. The results differ from the full-strip problem by shearing strain as shown in equation (41).

References

1. B. A. Boley, "A method for the construction of Green's functions", *Q. J. Appl. Math.*, **1956**, *14*, 249-257.
2. J. D. Humphrey and K. R. Rajagopol, "Finite thermoealsticity of constrained elastomers subject to biaxial loading", *J. Elasticity*, **1998**, *40*, 189-200.
3. M. D. Greenberg, "Application of Green's Functions in Science and Engineering", Prentice-Hall Inc., Englewood Cliffs, N. J., **1971**.
4. V. D. Seremet, "Handbook of Green's Functions and Matrices", WIT Press, Southampton, **2003**.
5. S. P. Gavelia, "On calculation of Green's matrices for applied problems allowing separation of variables", *J. Boundary Value Problems Math. Phys. Kiev*, **1979**, *13*, 59-67.
6. P. J. Udoh, "On the separation of variables in Green's functions for elastic materials", *Kenya J. Sci.*, **2010** (in press).
7. Y. A. Melnikov, "Green's Functions in Applied Mechanics", Computational Mechanics Publications, Southampton, **1995**, p. 267.