Full Paper

Application of separation of variables in Green’s function to typical half-strip problem for elastic material

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Abstract: This paper presents the method of separation of variables in Green’s function and its application to the problem of half-strip of elastic material. Unlike that of the full strip, the half-strip problem differs in specification of the boundary conditions. Though similar to the full-strip problem in conjugality conditions, the half-strip problem is amenable to the application of Poisson’s equation instead of Laplace’s equation. The results show that this problem is one-dimensional and the shearing stresses are non-symmetric.

Keywords: Green's function, half-strip problem, elastic material

Introduction

The method of separation of variables is the most general one for constructing Green’s functions. This method was used to solve a full-strip elastic problem [1]. The approach here differs from that of Humphrey and Rajagopol [2] where the basic idea is that of decomposing the motion into two parts: one due to traction-free uniform heating and the other to isothermal mechanical loading.

However, in the method of separation of variables, the approach is holistic without analogous representation into singular and regular parts [3]. The method of separation of variables enables us to construct the Green’s functions both for Laplace’s and Poisson’s operators [4] for simple boundary-value problems. This method allows us to represent the functions in an infinite series as is done in regularity conditions [5] and for the strain energy $W$ [2]. The method adopted here enables us to construct and take into account the property of orthogonality for trigonometric functions in terms of one-dimensional differential equations.
Mathematical Formulations

The Green’s function $G(x, \xi)$ for Poisson’s equation $\nabla^2_x G(x, \xi) = -\delta(x - \xi)$, and for the half-strip we specify the intervals as $0 < x_1 < \infty$ and $0 \leq x_2 \leq a_2$ under the following conditions:
$$\frac{\partial G}{\partial x_1} = 0 ; \ x_1 = 0, \ 0 \leq x_2 \leq a_2$$
$$\frac{\partial G}{\partial x_1} = 0 ; \ x_2 = 0, \ a_2 ; \ 0 \leq x_1 \leq \infty$$
Here, the function $G(x, \xi)$ should be bounded at infinity, i.e. $G \mid_{x_1=0} < \infty$

Computational Procedure

Following our earlier results [6], the solution to this problem is sought by means of the method of separation of variables leading to the following general trigonometric series:
$$G = a_0 + \sum_{n-1}^{n} a_n \cos v_1 x_2 + \sum_{m=1}^{n} b_m \sin v_1 x_2$$
where, $a_0$, $a_n$, and $b_m$ are the functions of the independent variable $x_i$. The boundary conditions of this problem with respect to the independent variables $x_2$ reduce equation (3) with the general series to the following form:
$$G = a_0 + \sum_{n=1}^{\infty} a_n \cos v_1 x_2, \ v_1 = \frac{n \pi}{a_2}, \ n = 1, 2, 3, \ldots \ldots$$
Substituting this expression in (4) for the Green’s function $G$ into the Poisson’s $\nabla^2_x G(x, \xi)$, we get
$$G = a_0^* + \sum_{n=1}^{\infty} (a_n^* - v_1^2 a_n) \cos v_1 x_2 = -\delta(x_1 - \xi_1) \delta(x_2 - \xi_2)$$
where, in the method of separation of variables, it takes the form:
$$\delta(x - \xi) = \delta(x_1 - \xi_1) \delta(x_2 - \xi_2)$$
Integrating equation (5) with respect to the independent variable $x_2$, we note that all the integrals with the exceptions of equations (7) and (8):
$$\int_{a_2}^{a_2} a_n^* \cos v_1 x_2 = a_n^* a_2$$
$$\int_{a_2}^{a_2} \delta(x_1 - \xi_1) \delta(x_2 - \xi_2) \ dx_2 = \delta(x_1 - \xi_1)$$
are equal to zero. Therefore, to determine the function $a_0(x_i)$, we obtain the following differential equation of the second order:
$$a_0^* = -a_2^{-1} \delta(x_1 - \xi_1)$$
and the boundary condition and the condition at infinity:
$$a_0^* (x_1 = 0) ; \ a_0 (x_1 = \infty) < \infty$$
To construct the Green’s function for this boundary-value problem, we use the standard technique [2]. The general solution of the 1D equation is sought in the form of
From the boundary condition and the condition at infinity, we obtain
\[ a_0'(x_1 = 0) = 0 \Rightarrow c_1 = 0; \quad a_0(\infty) = \infty \Rightarrow k_1 = 0 \] (12)

So the function \( a_0(x_1) \) can be rewritten in the form:
\[ a_2(x_1) = \begin{cases} c_2, & x_1 \leq \xi_1 \\ k_1, & x_1 \geq \xi_1 \end{cases} \] (13)

which satisfies the ordinary differential equation of the first order. Next, from the condition of conjugality at the point \( x_1 = \xi_1 \),
\[ a_0(x_1 = \xi_1 - 0) = a_0(x_1 = \xi_1 - +0) \] (14)

We obtain the following result: \( c_2 = k_2 = b = \text{constant} \).

Finally, for the desired function, we find
\[ a_0(x_1, \xi_1) = \begin{cases} b, & x_1 \leq \xi_1 \\ b, & x_1 \geq \xi_1 \end{cases} \] (15)

Now, to determine the function \( a_m \), we multiply both parts of the equation (5) by
\[ \cos v_2 x_2 \quad (v_2 = \frac{s\pi}{a_2}, \quad s = 1, 2, 3, \ldots) \] (16)

and take the integral with respect to the variable \( x_2 \).

Taking into account the property of orthogonality for trigonometric functions, we get the respective 1D differential equation:
\[ (a_n^2 - v_1^2 a_n) \frac{a_2}{2} = -\delta (x_1 - \xi_1) \cos v_1 \xi_2 \] (17)

and the boundary conditions:
\[ a_n'(x_1 = 0); \quad a_n'(x_1 = \infty) < \infty \] (18)

In solving equation (17), the following integrals have been calculated:
\[ \int_0^a \cos v_2 x_2 \cos v_1 x_2 d x_2 = \begin{cases} 0, & v_1 \neq v_2 \\ a_0 / 2, & v_1 = v_2 \end{cases} \] (19)

\[ \int_0^a \delta (x_1 - \xi_1) \delta (x_2 - \xi_2) \cos v_2 x_2 d x_2 = \delta (x_1 - \xi_1) \cos v_2 \xi_2 \] (20)

By taking a notation, \( a_m = 2a_2^{-1} a_\infty \cos v_2 \xi_2 \) (21)

with the conditions in (18), we reduce equation (19) to the boundary-value problem formulated to determine the function \( a_m \), i.e.
\[ (a_n^2 - v_1^2 a_n) = -\delta (x_1 - \xi_1) \] (22)
\[ a_n'(x_1 = 0) = 0 \] (23)
\[ a_n'(x_1 = \infty) < \infty \] (24)
Using the standard technique [7], the general solution of equation (11) is written in the form:

$$a_n = \begin{cases} c_1 e^{-v_1 x_1} + c_2 e^{v_1 x_1} ; & x_1 \leq \xi_1 \\ k_1 e^{-v_1 x_1} + k_2 e^{v_1 x_1} ; & x_1 \geq \xi_1 \end{cases}$$  \hspace{1cm} (25)$$

Then from the conditions of conjugality at the point $x_1 = \xi_1$,

$$a_n (x_1 = \xi_1 - 0) = a_n (x_1 = \xi_1 + 0)$$  \hspace{1cm} (26)$$

$$a_n (x_1 = \xi_1 - 0) - a_n (x_1 = \xi_1 + 0) = 1$$  \hspace{1cm} (27)$$

we get a system of two simultaneous linear algebraic equations:

$$(c_1 - k_1) e^{-v_1 \xi_1} + (c_2 - k_2) e^{v_1 \xi_1} = 0$$  \hspace{1cm} (28)$$

$$v_1 [ (c_1 - k_1) e^{-v_1 \xi_1} - (c_2 - k_2) e^{v_1 \xi_1} ] = -1$$  \hspace{1cm} (29)$$

Then from the boundary conditions and conditions at infinity, we get

$$a_m (x_1 = 0) = 0 \Rightarrow c_2 - c_1 = 0$$  \hspace{1cm} (30)$$

$$a_m' (x_1 = \infty) < \infty \Rightarrow k_2 = 0$$  \hspace{1cm} (31)$$

For the coefficients, we finally obtain the following values:

$$k_1 = \frac{e^{-v_1 \xi_1} - e^{v_1 \xi_1}}{2v_1}; \quad c_1 = c_2 = \frac{e^{-v_1 \xi_1}}{2v_1}; \quad k_2 = 0$$  \hspace{1cm} (32)$$

Using standard transformations, the Green’s functions in equation (25) can be represented as

$$a_n (x_1, \xi_1) = \begin{cases} \frac{1}{2v_1} \left(e^{v_1 (x_1 - \xi_1)} + e^{-v_1 (x_1 - \xi_1)} \right) ; & x_1 \leq \xi_1 \\ \frac{1}{2v_1} \left(e^{-v_1 (x_1 - \xi_1)} + e^{v_1 (x_1 - \xi_1)} \right) ; & x_1 \geq \xi_1 \end{cases}$$  \hspace{1cm} (33)$$

On account of equation (21), the Green’s function $G(x, \xi)$ for the initial boundary-value problem can be written in the form:

$$G(x, \xi) = \begin{cases} G_j (x, \xi) = b + \frac{2}{a_2} \sum_{n=1}^{\infty} \frac{1}{2v_1} (e^{v_1 (x_1 - \xi_1)} + e^{-v_1 (x_1 - \xi_1)}) \cos v_1 x_2 \cos v_1 \xi_2 ; & x_1 \leq \xi_1 \\ G_j (x, \xi) = b + \frac{2}{a_2} \sum_{n=1}^{\infty} \frac{1}{2v_1} (e^{v_1 (x_1 - \xi_1)} + e^{-v_1 (x_1 - \xi_1)}) \cos v_1 x_2 \cos v_1 \xi_2 ; & x_1 \geq \xi_1 \end{cases}$$  \hspace{1cm} (34)$$

However, by making use of the known sum [1, 3, 4],

$$\sum_{n=1}^{\infty} \frac{p^n}{n} \cos n\alpha = -\ln \sqrt{1 - 2p \cos \alpha + p^2} ; p^2 < 1, \quad 0 \leq \alpha \leq 2\pi; \quad \text{or} \quad p^2 \leq 1, \quad 0 \leq \alpha \leq 2\pi$$  \hspace{1cm} (35)$$

and trigonometric formula:

$$\cos v_1 x_2 \cos v_1 \xi_2 = \frac{1}{2} \left[ \cos v_1 (x_1 - \xi_2) + \cos v_1 (x_2 + \xi_2) \right]$$  \hspace{1cm} (36)$$

then one can take the sum of the ordinary infinite series. After some computation, we obtain the final expression for the Green’s function of the initial boundary-value Neumann’s problem for the half-strip as:

$$G = b - \frac{1}{2\pi} \ln EE_1 E_2 E_{12}$$  \hspace{1cm} (37)$$

where the functions $E_1, E_2, E_{12}$ are determined by the expressions:
\[ E = \sqrt{1 - 2e^{\alpha_2^2(x_1 - \xi_1)}} \cos \pi/a_2 \left( x_2 - \xi_2 \right) + e^{\alpha_2^2(x_1 - \xi_1)} \]  
(38)

\[ E_1 = \sqrt{1 - 2e^{-\alpha_2^2(x_1 + \xi_1)}} \cos \pi/a_2 \left( x_2 - \xi_2 \right) + e^{\alpha_2^2(x_1 + \xi_1)} \]  
(39)

\[ E_2 = \sqrt{1 - 2e^{\alpha_2^2(x_1 - \xi_1)}} \cos \pi/a_2 \left( x_2 + \xi_2 \right) + e^{\alpha_2^2(x_1 - \xi_1)} \]  
(40)

\[ E_{12} = \sqrt{1 - 2e^{-\alpha_2^2(x_1 + \xi_1)}} \cos \pi/a_2 \left( x_2 + \xi_2 \right) + e^{\alpha_2^2(x_1 + \xi_1)} \]  
(41)

Conclusions

In this paper, we have made use of the method of separation of variables in Green’s functions to solve the problem of the half-strip problem. A particular focus is given to the problem of the half-strip using Poisson’s equation. For ease of formulation, we use trigonometric functions to construct the Green’s functions for the Poisson’s equation. The results differ from the full-strip problem by shearing strain as shown in equation (41).

References


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