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Full Paper

New exact travelling wave solutions of generalised sinh-Gordon and (2 + 1)-dimensional ZK-BBM equations

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Abstract: Exact travelling wave solutions have been established for generalised sinh-Gordon and generalised (2+1) dimensional ZK-BBM equations by using $\left(\frac{G'}{G}\right)-$expansion method where $G = G(\xi)$ satisfies a second-order linear ordinary differential equation. The travelling wave solutions are expressed by hyperbolic, trigonometric and rational functions.

Keywords: $\left(\frac{G'}{G}\right)-$expansion method, travelling wave solutions, generalised sinh-Gordon equation, (2+1) dimensional ZK-BBM equation

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INTRODUCTION

Non-linear partial differential equations (PDEs) are widely used as models to describe complex physical phenomena in various fields of sciences, especially fluid mechanics, solid state physics, plasma physics, plasma wave and chemical physics. Particularly, various methods have been utilised to explore different kinds of solutions of physical models described by non-linear PDEs. One of the basic physical problems for these models is to obtain their exact solutions. In recent years various methods for obtaining exact travelling wave solutions to non-linear equations have been presented, such as the homogeneous balance method [1], the tanh function method [2, 3], the Jacobi elliptic function method [4, 5] and the F-expansion method [6, 7].

In this paper, we use $\left(\frac{G'}{G}\right)-$expansion method [8, 9] to establish exact travelling wave solutions for generalised sinh-Gordon and generalised (2+1) dimensional ZK-BBM equations. The main idea of this method is that the travelling wave solutions of non-linear equations can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$, where $G = G(\xi)$ satisfies the second-order linear ordinary differential equation (LODE): $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where $\xi = x - ct$ and λ, μ and c are arbitrary constants. The degree of this polynomial can be determined by considering the homogeneous balance between higher-order derivatives and the non-linear term appearing in the given non-linear equations. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulting from the process of using the proposed method.

The sinh-Gordon equation, viz.

$$u_{xt} = \sinh(u), \tag{1}$$

appears in many branches of non-linear science and plays an important role in physics [10], being an important model equation studied by several authors [11–14].

Wazwaz [14] studied the generalised sinh-Gordon equation given by

$$u_{tt} - au_{xx} + b\sinh(nu) = 0, \tag{2}$$

where a and b are arbitrary constants and n is a positive integer. By using the tanh method, Wazwaz derived exact travelling wave solutions of Eq.(2), which provides a more powerful model than the Eq.(1).

The generalised form of the (2 + 1) dimensional ZK-BBM equation is given as:

$$u_t + u_x - a(u^2)_x + (bu_{xt} - ku_{yt})_x = 0, \tag{3}$$

where a, b and k are arbitrary constants. It arises as a description of gravity water waves in the long-wave regime [15, 16]. A variety of exact solutions for the $(2+1)$ dimensional ZK-BBM equation [17, 18] are developed by means of the tanh and the sine-cosine methods. In this paper we construct new travelling wave solutions of Eqs.(2) and (3).

(*G′/G*)-EXPANSION METHOD

In this section we describe the $\left(\frac{G'}{G}\right)-$expansion method [8, 9, 19] to find the travelling wave solutions of non-linear PDEs. Suppose that a non-linear equation with two independent variables x and t is given by

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots\ldots\ldots\ldots\ldots) = 0, \tag{4}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and non-linear terms are involved. In the following the main steps of the $\left(\frac{G'}{G}\right)-$expansion method are given.

Step 1. Combining the independent variables x and t into one variable $\xi = x - Vt$,

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we suppose that

$$u(x,t) = u(\xi),\ \xi = x - Vt. \tag{5}$$

The travelling wave variables (5) permits us to reduce Eq.(4) to an ordinary differential equation (ODE):

$$P(u, -Vu', u', V^2u'', -Vu'', u'', ...) = 0. \tag{6}$$

Step 2. Suppose that the solution of the ODE (6) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$u(\xi) = \alpha_m (\frac{G'}{G})^m + ..., \qquad \alpha_m \neq 0, \tag{7}$$

where $G = G(\xi)$ satisfies the second-order LODE in the following form:

$$G'' + \lambda G' + \mu G = 0. \tag{8}$$

The constants α_m , $\alpha_{m-1} ... \lambda$ and μ are to be determined later. The unwritten part in Eq.(7) is also a polynomial in $\left(\frac{G'}{G}\right)$, the degree of which is, however, generally equal to or less than *m*-1. The positive integer m can be determined by considering the homogenous balance between the highest order derivatives and the non-linear terms appearing in ODE (6).

Step 3. By substituting (7) into Eq.(6) and using second-order LODE (8), the left-hand side of Eq.(6) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $\alpha_m, ..., V, \lambda, \mu$.

Step 4. The constants $\alpha_m, ..., V, \lambda, \mu$ can be obtained by solving the system of algebraic equations obtained in Step 3. Since the general solutions of the second-order LODE (8) is well known depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, the exact solutions of the given Eq.(4) can be obtained.

GENERALISED SINH-GORDON EQUATION

To find the explicit exact solutions of the sinh-Gordon Eq.(2), we proceed with the methodology explained in the above section. First we make the transformation:

$$u(x,t) = u(\xi) = u(x - ct), \tag{9}$$

where *c* is the wave speed. Substituting the above travelling wave transformation into (2), we get the following ODE:

$$du_{\xi\xi} + b\sinh(nu) = 0, \tag{10}$$

where $d = c^2 - a$ and b is arbitrary constant. Appling the transformation $v = e^{nu}$ to Eq.(10) and using relations

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$$\sinh(nu) = \frac{v - v^{-1}}{2},\ \cosh(nu) = \frac{v + v^{-1}}{2},\ u = \frac{1}{n}\operatorname{arccos}h\frac{v + v^{-1}}{2},\ u'' = \frac{v''v - v'^2}{nv^2}$$

in Eq.(10), we have

$$2d(v''v - v'^2) + bnv^3 - bnv = 0. \tag{11}$$

Suppose that the solution of ODE (11) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$v(\xi) = \alpha_m (\frac{G'}{G})^m + ..., \tag{12}$$

where $G(\xi)$ satisfies the second-order LODE in the form:

$$G'' + \lambda G' + \mu G = 0\,. \tag{13}$$

From Eqs.(12) and (13), we have

$$\begin{aligned} v^3 &= \alpha_m{}^3 (\frac{G'}{G})^{3m} + ... \\ v' &= -m\alpha_m (\frac{G'}{G})^{m+1} + ... \\ v'' &= m(m+1)\alpha_m (\frac{G'}{G})^{m+2} + ... \end{aligned} \tag{14}$$

Considering the homogeneous balance between $v''v$ and v^3 in Eq.(11), it is required that $m = 2$. So we can write (12) as:

$$v(\xi) = \alpha_2 (\frac{G'}{G})^2 + \alpha_1 (\frac{G'}{G}) \ + \alpha_0;\ \alpha_2 \neq 0\,. \tag{15}$$

By using Eq.(15) and (13) in Eq.(11) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left-hand side of Eq.(11) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for $\alpha_{1,}\alpha_2,\alpha_0$ and d as follows:

$$4d\alpha_2^2 + bn\alpha_2^3 = 0$$

$$8d(-2\alpha_2\lambda - \alpha_1)\alpha_2 + 3bn\alpha_1\alpha_2^2 + 2d(2\alpha_1 + 10\alpha_2\lambda)\alpha_2 + 12d\alpha_2\alpha_1 = 0$$

$$bn\left(\alpha_0\alpha_2^2 + 2\alpha_1^2\alpha_2 + \alpha_2 + \alpha_2(2\alpha_0\alpha_2 + \alpha_1^2)\right) + 2d(8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2) + 2d(2\alpha_1 + 10\alpha_2\lambda)\alpha_1 + 12d\alpha_2\alpha_0$$
$$-2d(-4(-2\alpha_2\mu - \alpha_1\lambda)\alpha_2 + (-2\alpha_2\lambda - \alpha_1)^2) = 0$$

$$bn(4\alpha_1\alpha_2\alpha_0 + \alpha_1(2\alpha_0\alpha_2 + \alpha_1^2)) + 2d(6\alpha_2\mu\lambda + 2\alpha_1\mu + \alpha_1\lambda^2)\alpha_2 + 2d(8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)\alpha_1$$
$$+2d(2\alpha_1 + 10\alpha_2\lambda)\alpha_0 - 2d(4\alpha_1\mu\alpha_2 + 2(-2\alpha_2\mu - \alpha_1\lambda)(-2\alpha_2\lambda - \alpha_1)) = 0$$

$$bn(\alpha_0(2\alpha_0\alpha_1 + \alpha_1^2) + 2\alpha_1^2\alpha_0 + \alpha_2\alpha_0^2) - bn\alpha_2 + 2d(2\alpha_2\mu^2 + \alpha_1\lambda\mu)\alpha_2 + 2d(6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)\alpha_1$$
$$-2d(8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)\alpha_0 - 2d(-2\alpha_1\mu(-2\alpha_2\lambda - \alpha_1) + (-2\alpha_2\mu - \alpha_1\lambda)^2) = 0$$

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$$3bn\alpha_0^2\alpha_1 - bn\alpha_1 + 2d(2\alpha_2\mu^2 + \alpha_1\lambda\mu)\alpha_1 + 2d(6\alpha_2\lambda\mu + 2\alpha_1\mu + \alpha_1\lambda^2)\alpha_0 + 4d\alpha_1\mu(-2\alpha_2\mu - \alpha_1\lambda) = 0$$
$$2d(2\alpha_2\mu^2 + \alpha_1\lambda\mu)\alpha_0 + bn\alpha_0^3 - 2d\alpha_1^2\mu^2 - bn\alpha_0 = 0 \,. \tag{16}$$

Solving the algebraic equations above yields

$$\alpha_0 = \pm\frac{\lambda^2}{4\mu-\lambda^2},\ \alpha_1 = \pm\frac{4\lambda}{4\mu-\lambda^2},\ \alpha_2 = \pm\frac{4}{4\mu-\lambda^2},\ d = \mp\frac{6n}{4\mu-\lambda^2}, \tag{17}$$

where λ, μ and n are arbitrary constants.

By using (17), expression (15) can be written as:

$$v(\xi) = \pm\frac{4}{4\mu-\lambda^2}\left(\frac{G'}{G}\right)^2 \pm \frac{4\lambda}{4\mu-\lambda^2}\left(\frac{G'}{G}\right) \pm \frac{\lambda^2}{4\mu-\lambda^2}, \tag{18}$$

where $\xi = x - t\sqrt{\mp\frac{6n}{4\mu-\lambda^2} + a}$.

Substituting the general solution of Eq.(13) into Eq.(18), we have two types of travelling wave solution of Eq.(2) as follows:

Case 1: When $\lambda^2 - 4\mu < 0$,

$$u(x,t) = \frac{1}{n}\log\left(\pm\frac{4}{4\mu-\lambda^2}\left(\frac{G'}{G}\right)^2 \pm \frac{4\lambda}{4\mu-\lambda^2}\left(\frac{G'}{G}\right) \pm \frac{\lambda^2}{4\mu-\lambda^2}\right), \tag{19}$$

where $\left(\frac{G'}{G}\right) = -\frac{\lambda}{2} + \frac{1}{2}\frac{\left(C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)\right)\sqrt{\lambda^2-4\mu}\xi}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}$ and

$\xi = x - t\sqrt{\mp\frac{6n}{4\mu-\lambda^2} + a}$. C_1 and C_2 are arbitrary constants. Profile of solution (19) for $n = a = C_2 = 1, C_1 = \mu = 0$ and $\lambda = \sqrt{2}$ is shown in Figure 1.

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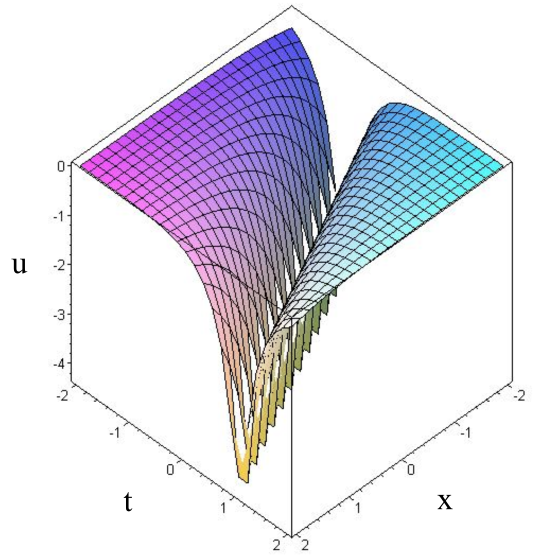

Figure 1. Profile of solution (19) when $n = a = C_2 = 1, C_1 = \mu = 0$ and $\lambda = \sqrt{2}$

Case 2: When $\lambda^2 - 4\mu > 0$,

$$u(x,t) = \frac{1}{n}\log\left(\pm\frac{4}{4\mu-\lambda^2}(\frac{G'}{G})^2 \pm \frac{4\lambda}{4\mu-\lambda^2}(\frac{G'}{G}) \pm \frac{\lambda^2}{4\mu-\lambda^2} \right), \tag{20}$$

where $\left(\frac{G'}{G}\right) = -\frac{\lambda}{2} + \frac{1}{2}\frac{\left(-C_1 \sin\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \cos\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) \right)\sqrt{\lambda^2-4\mu}\xi}{C_1 \cos\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right) + C_2 \sin\left(\frac{\sqrt{\lambda^2-4\mu}\xi}{2}\right)}$ and

$\xi = x - t\sqrt{\mp\frac{6n}{4\mu-\lambda^2} + a}$. C_1 and C_2 are arbitrary constants. In Figure 2 we have periodic solution (20) when $n = a = \lambda = C_1 = C_2 = 1$ and $\mu = 2\sqrt{2}$.

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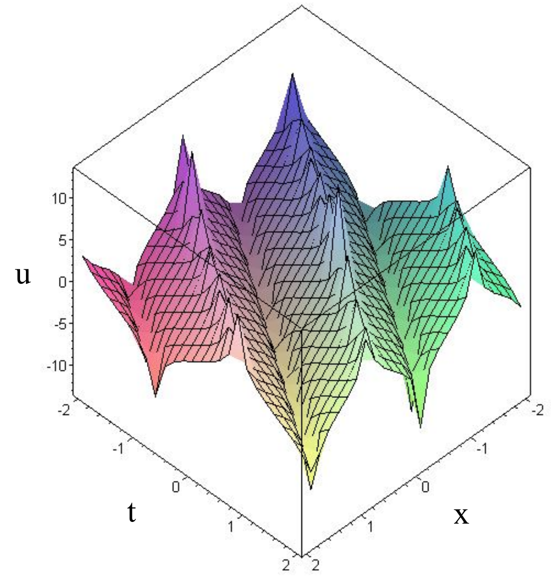

Figure 2. 3D plot of periodic solution (20) when $n = a = \lambda = C_1 = C_2 = 1$ and $\mu = 2\sqrt{2}$

(2+1) DIMENSIONAL ZK-BBM EQUATION

In finding exact solutions of ZK-BBM Eq.(3), first we make the transformation:

$$u(\xi) = u(\xi),\ \xi = x + y - ct, \tag{21}$$

which permits us to convert Eq.(3) into an ODE for $u = u(\xi)$ as follows:

$$u'(1-c) - 2auu' + cu'''(b-k) = 0, \tag{22}$$

where a, b and k are arbitrary parameters and c is the wave speed.

Integrating it with respect to ξ once yields:

$$V + u(1-c) - au^2 + cu''(b-k) = 0, \tag{23}$$

where V is an integration constant that is to be determined later.

Proceeding in a similar manner as in the above section and considering the homogeneous balance between u'' and u^2 in Eq.(23), we have $m = 2$. So we can assume the solution of Eq.(23) as follows:

$$u(\xi) = \alpha_2(\frac{G'}{G})^2 + \alpha_1(\frac{G'}{G}) + \alpha_0;\quad \alpha_2 \neq 0, \tag{24}$$

where α_0, α_1 and α_2 are to be determined later.

By using (24) and (13) in Eq.(23) and collecting all terms with the same power of $\left(\frac{G'}{G}\right)$ together, the left-hand side of Eq.(23) is converted into another polynomial in $\left(\frac{G'}{G}\right)$. Equating each

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coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for c and V as follows:

$$\begin{aligned}
&-a\alpha_2^2 + bc\alpha_2(b-k) = 0\\
&c(2\alpha_1 + 10\alpha_2\lambda)(b-k) - 2a\alpha_1\alpha_2 = 0\\
&\alpha_2(1-c) + c(8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2)(b-k) - a(2\alpha_0\alpha_2 + \alpha_1^2) = 0\\
&\alpha_1(1-c) + c(b\alpha_2\lambda\mu + 2\alpha_1\lambda + 4\alpha_2\lambda^2)(b-k) - 2a\alpha_0\alpha_1 = 0\\
&V + \alpha_0(1-c) - a\alpha_0^2 + c(2\alpha_2\mu^2\alpha_1\lambda\mu)(b-k) = 0.
\end{aligned} \tag{25}$$

Solving the algebraic equations above yields

$$\alpha_2 = \frac{bc(b-k)}{a}, \alpha_1 = \frac{b\lambda c(b-k)}{a}, \alpha_0 = \frac{b\lambda^2 c + 8bc\mu - k\lambda^2 c + 1 - c - 8kc\mu}{2a},$$
$$V = \frac{\begin{array}{l}(2c - 8b^2\lambda^2c^2\mu - 8k^2\lambda^2c^2\mu + 16b\lambda^2c^2\mu k + 16c^2k^2\mu^2 + 16c^2k^2\mu^2 - 32bc^2k\mu^2\\ -2bkc^2\lambda^4 + b^2\lambda^4c^2 + k^2\lambda^4c^2 - 1 - c^2)\end{array}}{4a}, \tag{26}$$

where λ, μ, a, b and k are arbitrary constants.

By using (26), expression (24) can be written as:

$$u(\xi) = \frac{bc(b-k)}{a}\left(\frac{G'}{G}\right)^2 + \frac{b\lambda c(b-k)}{a}\left(\frac{G'}{G}\right) + \frac{b\lambda^2 c + 8bc\mu - k\lambda^2 c + 1 - c - 8kc\mu}{2a} \tag{27}$$

where λ, μ, b, c and k are arbitrary constants.

Substituting the general solution of Eq.(13) into Eq.(27), we have the following travelling wave solutions of Eq. (3):

Case 1. When $\lambda^2 - 4\mu > 0$,

$$u(\xi) = \frac{bc(b-k)}{a}\left(\frac{G'}{G}\right)^2 + \frac{b\lambda c(b-k)}{a}\left(\frac{G'}{G}\right) + \frac{b\lambda^2 c + 8bc\mu - k\lambda^2 c + 1 - c - 8kc\mu}{2a}, \tag{28}$$

where $\left(\frac{G'}{G}\right) = -\frac{\lambda}{2} + \frac{1}{2}\frac{\left(C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right)\right)\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}\xi}{2}\right)}$

and $\xi = x + y - ct$. C_1 and C_2 are arbitrary constants. The solution (28) furnishes a bell-shaped soliton solution (28) when $a = k = c = C_1 = 1, b = 2, t = 0.2, C_2 = \mu = 0$ and $\lambda = \sqrt{2}$, as shown in Figure 3.

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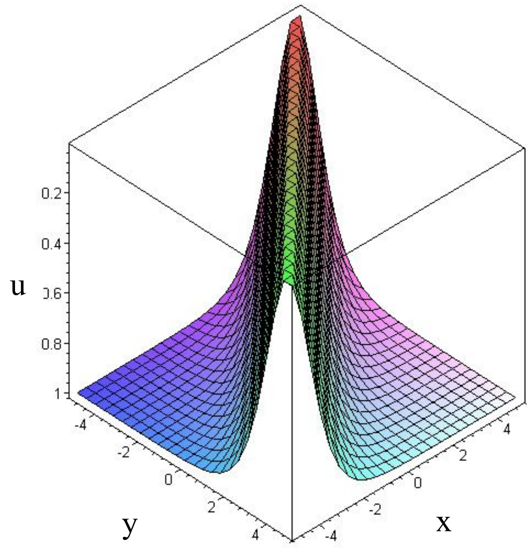

Figure 3. Bell-shaped soliton solution (28) when $a = k = c = C_1 = 1, b = 2, t = 0.2, C_2 = \mu = 0$ and $\lambda = \sqrt{2}$

Case 2. When $\lambda^2 - 4\mu < 0$,

$$u(\xi) = \frac{bc(b-k)}{a}(\frac{G'}{G})^2 + \frac{b\lambda c(b-k)}{a}(\frac{G'}{G}) + \frac{b\lambda^2 c + 8bc\mu - k\lambda^2 c + 1 - c - 8kc\mu}{2a}, \tag{29}$$

where $\frac{\mathrm{G}'(\xi)}{\mathrm{G}(\xi)} = -\frac{1}{2}\lambda + \frac{1}{2}\frac{\left(-C_1 \sin(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi) + C_2 \cos(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)\right)\sqrt{-\lambda^2+4\mu}}{C_1 \cos(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)_2 \sin(\frac{1}{2}\sqrt{-\lambda^2+4\mu}\xi)}$

and $\xi = x + y - ct$. C_1 and C_2 are arbitrary constants. For $a = 1.5$, $k = c = C_1 = 1$, $b = 2$, $t = 0.2$, $\lambda = C_2 = 0$ and $\mu = 1$, solution (29) gives the periodic solution as shown in Figure 4.

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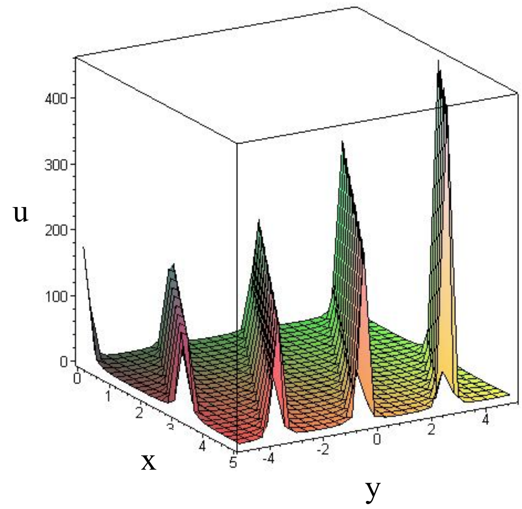

Figure 4. Periodic solution (29) when $a = 1.5, k = c = C_1 = 1, b = 2, t = 0.2, \lambda = C_2 = 0$ and $\mu = 1$

Case 3. When $\lambda^2 - 4\mu = 0$,

$$u(\xi) = \frac{bc(b-k)}{a}\left(\frac{2C_2 - C_1\lambda - C_2\lambda\xi}{2(C_1 + C_2\xi)}\right)^2 + \frac{b\lambda c(b-k)}{a}\left(\frac{2C_2 - C_1\lambda - C_2\lambda\xi}{2(C_1 + C_2\xi)}\right) + \frac{b\lambda^2 c + 8bc\mu - k\lambda^2 c + 1 - c - 8kc\mu}{2a}, \tag{30}$$

where $\xi = x + y - ct$. C_1 and C_2 are arbitrary constants. In Figure 5, we have a soliton solution (30) for $a = 1.5, k = c = C_1 = C_2 = 1, b = 2, t = 0.2, \lambda = 1$ and $\mu = \frac{1}{4}$.

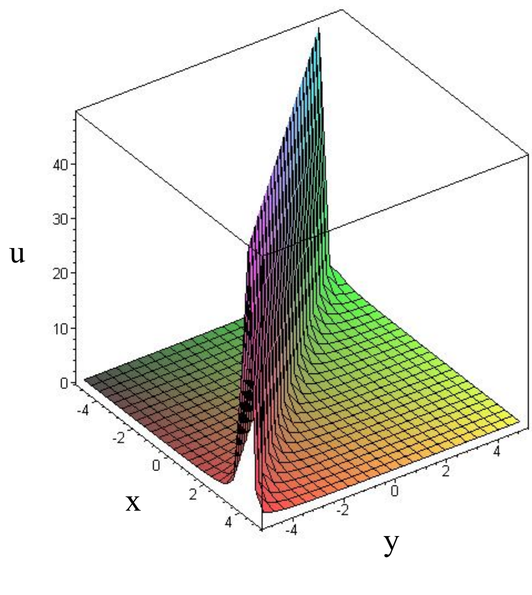

Figure 5. Soliton solution (30) for $a = 1.5, k = c = C_1 = C_2 = 1, b = 2, t = 0.2, \lambda = 1$ and $\mu = \frac{1}{4}$

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CONCLUSIONS

In this paper, the travelling wave solutions of the generalised sinh-Gordon and (2 + 1) dimensional ZK-BBM equations are found successfully through the use of $\left(\frac{G'}{G}\right)$-expansion method, which includes hyperbolic function solutions and trigonometric function solutions. One can see that this method is direct, concise and effective.

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