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*Full Paper*

# Non-differentiable second-order mixed symmetric duality with cone constraints

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**Abstract:** A pair of mixed non-differentiable second-order symmetric dual programmes over cones is considered. Weak, strong and converse duality theorems are established under second-order $(F, \rho)$ convexity/pseudo-convexity assumptions. Special cases are also discussed to show that this paper extends some known results in the literature.

**Keywords:** symmetric duality, mixed second-order duality, second-order $(F, \rho)$ convexity/pseudo-convexity, non-differentiable programming, cone constraints

## INTRODUCTION

The study of symmetric duality for non-linear problems is well developed by many researchers, notably Dantzig et al. [1], Mond [2], Bazaraa and Goode [3], and Mond and Weir [4]. Mangasarian [5] was the first to identify second-order dual formulations for the non-linear programmes. Wolfe type second-order symmetric duality has been discussed by Ahmad and Husain [6] and Yang et al. [7] for single-objective non-differentiable functions, and by Yang et al. [8] and Ahmad and Husain [9] for multi-objective programming problems. The duality results for a pair of Mond-Weir type second-order multi-objective symmetric dual programmes have been considered by Suneja et al. [10] under $\eta$-bonvexity/$\eta$-pseudo-bonvexity assumptions. Ahmad and Husain [9, 11], Khurana [12], Mishra and Lai [13], and Padhan and Nahak [14] studied symmetric duality over arbitrary cones.

Bector et al. [15] and Yang et al. [16] formulated mixed symmetric dual models for multi-objective differentiable and single-objective non-differentiable programming problems respectively. Ahmad [17] introduced a mixed-type symmetric duality in multi-objective programming problems,

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ignoring non-negativity restrictions of Bector et al. [15]. Recently, Ahmad and Husain [11] studied duality results for a pair of multi-objective mixed symmetric dual programmes over arbitrary cones under *K*-preinvexity/*K*-pseudo-invexity assumptions. Li and Gao [18] obtained duality relations for a mixed symmetric dual model in non-differentiable multi-objective non-linear programming problems involving generalised convex functions. Recently, Gupta and Kailey [19] formulated a second-order mixed symmetric dual programme for a class of non-differentiable multi-objective programming problems and proved usual duality results under second-order *F*-convexity/pseudo-convexity assumptions.

In this paper, we formulate a pair of mixed non-differentiable second-order symmetric dual programmes over arbitrary cones. Weak, strong and converse duality results are proved under second-order $(F,\rho)$ convexity/pseudo-convexity assumptions. Several known results are obtained as special cases.

## NOTATIONS AND PRELIMINARIES

**Definition 1** [11, 12, 20]**.** Let $C$ be a closed convex cone in $R^n$ with non-empty interior. The positive polar cone $C^*$ of $C$ is defined as

$$C^* = \{z : x^T z \geq 0, \text{for all } x \in C\}.$$

Now we consider a sub-linear functional $F : X \times X \times R^n \to R$ (where $X \subseteq R^n$).

**Definition 2.** A twicely differentiable function $\psi : X \to R$ is said to be second-order $(F,\rho)$ convex at $u \in X$, if there exists a real-value function $d : X \times X \to R$ and a real number $\rho$, such that for all $q \in R^n$ and $x \in X$,

$$\psi(x) - \psi(u) + \frac{1}{2} q^T \nabla_{xx}\psi(u) q \geq F\big(x,u; \nabla_x \psi(u) + \nabla_{xx}\psi(u) q\big) + \rho d^2(x,u).$$

**Definition 3.** A twicely differentiable function $\psi : X \to R$ is said to be second-order $(F,\rho)$ pseudo-convex at $u \in X$, if there exists a real-value function $d : X \times X \to R$ and a real number $\rho$, such that for all $q \in R^n$ and $x \in X$,

$$F\big(x,u; \nabla_x \psi(u) + \nabla_{xx}\psi(u) q\big) \geq 0 \Rightarrow \psi(x) \geq \psi(u) - \frac{1}{2} q^T \nabla_{xx}\psi(u) q + \rho d^2(x,u).$$

### Generalised Schwartz Inequality

The following generalised schwartz inequality shall be made use of:

$$l^T Am \leq (l^T Al)^{\frac{1}{2}} (m^T Am)^{\frac{1}{2}},$$

where $l, m \in R^n$, and $A \in R^n \times R^n$ is a positive semi-definite matrix. Equality holds if for some $\lambda \geq 0$, $Al = \lambda Am$.

## PROBLEM FORMULATION

For $N = \{1,2,\ldots,n\}$ and $M = \{1,2,\ldots,m\}$, let $J_1 \subseteq N, K_1 \subseteq M$ , $J_2 = N \setminus J_1$ and $K_2 = M \setminus K_1$. Let $|J_1|$ denote the number of elements in $J_1$. The other symbols, $|J_2|$,$|K_1|$ and $|K_2|$, are defined similarly. Let $x^1 \in R^{|J_1|}$ and $x^2 \in R^{|J_2|}$. Then any $x \in R^n$ can be written as $(x^1, x^2)$.

Similarly for $y^1 \in R^{|K_1|}$ and $y^2 \in R^{|K_2|}, y \in R^m$ can be written as $(y^1, y^2)$. It may be noted here that if $J_1 = \varnothing$, then $|J_1| = 0, J_2 = N$ and therefore $|J_2| = n$. In this case, $R^{|J_1|}, R^{|J_2|}$ and $R^{|J_1|} \times R^{|K_1|}$ will be zero-dimensional, *n*-dimensional and $|K_1|$-dimensional Euclidean spaces respectively. Similarly we can describe the cases $J_2 = \varnothing, K_1 = \varnothing$ or $K_2 = \varnothing$. Let $C_1$, $C_2$, $C_3$ and $C_4$ be closed convex cones with non-empty interiors in $R^{|J_1|}$, $R^{|J_2|}$, $R^{|K_1|}$ and $R^{|K_2|}$ respectively.

Now, consider the following pair of mixed second-order symmetric dual programs:

**Primal Problem (SMP)**

$$\text{Minimise } L(x^1, y^1, x^2, y^2, z^2, p, r) = f(x^1, y^1) + ((x^1)^T D_1 x^1)^{\frac{1}{2}} + g(x^2, y^2) + ((x^2)^T D_2 x^2)^{\frac{1}{2}}$$
$$-(y^2)^T E_2 z^2 - (y^1)^T [\nabla_{y^1} f(x^1, y^1) + \nabla_{y^1 y^1} f(x^1, y^1) p] - \frac{1}{2} p^T \nabla_{y^1 y^1} f(x^1, y^1) p$$
$$-\frac{1}{2} r^T \nabla_{y^2 y^2} g(x^2, y^2) r,$$

subject to

$$-[\nabla_{y^1} f(x^1, y^1) - E_1 z^1 + \nabla_{y^1 y^1} f(x^1, y^1) p] \in C_3^*, \tag{1}$$

$$-[\nabla_{y^2} g(x^2, y^2) - E_2 z^2 + \nabla_{y^2 y^2} g(x^2, y^2) r] \in C_4^*, \tag{2}$$

$$(y^2)^T [\nabla_{y^2} g(x^2, y^2) - E_2 z^2 + \nabla_{y^2 y^2} g(x^2, y^2) r] \geq 0, \tag{3}$$

$$(z^1)^T E_1 z^1 \leq 1, \tag{4}$$

$$(z^2)^T E_2 z^2 \leq 1, \tag{5}$$

$$x^1 \in C_1, x^2 \in C_2. \tag{6}$$

**Dual Problem (SMD)**

$$\text{Maximise } M(u^1, v^1, u^2, v^2, w^2, q, s) = f(u^1, v^1) - ((v^1)^T E_1 v^1)^{\frac{1}{2}} + g(u^2, v^2) - ((v^2)^T E_2 v^2)^{\frac{1}{2}}$$
$$+(u^2)^T D_2 w^2 - (u^1)^T [\nabla_{x^1} f(u^1, v^1) + \nabla_{x^1 x^1} f(u^1, v^1) q] - \frac{1}{2} q^T \nabla_{x^1 x^1} f(u^1, v^1) q$$
$$-\frac{1}{2} s^T \nabla_{x^2 x^2} g(u^2, v^2) s,$$

subject to

$$\nabla_{x^1} f(u^1, v^1) + D_1 w^1 + \nabla_{x^1 x^1} f(u^1, v^1) q \in C_1^*, \tag{7}$$

$$\nabla_{x^2} g(u^2, v^2) + D_2 w^2 + \nabla_{x^2 x^2} g(u^2, v^2) s \in C_2^*, \tag{8}$$

$$(u^2)^T [\nabla_{x^2} g(u^2, v^2) + D_2 w^2 + \nabla_{x^2 x^2} g(u^2, v^2) s] \leq 0, \tag{9}$$

$$(w^1)^T D_1 w^1 \leq 1, \tag{10}$$

$$(w^2)^T D_2 w^2 \le 1, \tag{11}$$

$$v^1 \in C_3, v^2 \in C_4. \tag{12}$$

where

1. $f : R^{|J_1|} \times R^{|K_1|} \to R$ and $g : R^{|J_2|} \times R^{|K_2|} \to R$ are differentiable functions,
2. $D_1$, $D_2$, $E_1$ and $E_2$ are positive semi-definite matrices in $R^{|J_1|} \times R^{|J_1|}, R^{|J_2|} \times R^{|J_2|}, R^{|K_1|} \times R^{|K_1|}$ and $R^{|K_2|} \times R^{|K_2|}$ respectively, and
3. $p, z^1 \in R^{|K_1|}, r, z^2 \in R^{|K_2|}, q, w^1 \in R^{|J_1|}$ and $s, w^2 \in R^{|J_2|}$.

## RESULTS AND DISCUSSION

In this section, we prove weak, strong and converse duality results for the dual pair, SMP and SMD, formulated above.

**Theorem 1 (Weak duality)**

Let $(x^1, y^1, x^2, y^2, z^1, z^2, p, r)$ be feasible for (SMP) and $(u^1, v^1, u^2, v^2, w^1, w^2, q, s)$ be feasible for (SMD). Let the sublinear functionals $F_1 : R^{|J_1|} \times R^{|J_1|} \times R^{|J_1|} \mapsto R$, $F_2 : R^{|K_1|} \times R^{|K_1|} \times R^{|K_1|} \mapsto R$, $G_1 : R^{|J_2|} \times R^{|J_2|} \times R^{|J_2|} \mapsto R$ and $G_2 : R^{|K_2|} \times R^{|K_2|} \times R^{|K_2|} \mapsto R$ satisfy the following conditions:

$$F_1(x^1, u^1; a^1) + (a^1)^T u^1 \ge 0, \text{ for all } a^1 \in C_1^*, \tag{A}$$

$$F_2(v^1, y^1; a^2) + (a^2)^T y^1 \ge 0, \text{ for all } a^2 \in C_3^*, \tag{B}$$

$$G_1(x^2, u^2; b^1) + (b^1)^T u^2 \ge 0, \text{ for all } b^1 \in C_2^*, \tag{C}$$

$$G_2(v^2, y^2; b^2) + (b^2)^T y^2 \ge 0, \text{ for all } b^2 \in C_4^*. \tag{D}$$

Further, let

*(i)* $f(., v^1) + (.)^T D_1 w^1$ be second-order $(F_1, \rho_1)$ convex at $u^1$, and $-(f(x^1, .) - (.)^T E_1 z^1)$ be second-order $(F_2, \rho_2)$ convex at $y^1$,

*(ii)* $g(., v^2) + (.)^T D_2 w^2$ be second-order $(G_1, \sigma_1)$ pseudo-convex at $u^2$, and $-(g(x^2, .) - (.)^T E_2 z^2)$ be second-order $(G_2, \sigma_2)$ pseudo-convex at $y^2$,

*(iii)* either $\rho_1 d_1^2(x^1, u^1) + \rho_2 d_2^2(v^1, y^1) \ge 0$ or $\rho_1, \rho_2 \ge 0$, and

*(iv)* either $\sigma_1 d_3^2(x^2, u^2) + \sigma_2 d_4^2(v^2, y^2) \ge 0$ or $\sigma_1, \sigma_2 \ge 0$.

Then

$$L(x^1, y^1, x^2, y^2, z^2, p, r) \ \ge \ M(u^1, v^1, u^2, v^2, w^2, q, s).$$

**Proof.** By the second-order $(F_1, \rho_1)$ convexity of $f(., v^1) + (.)^T D_1 w^1$ at $u^1$ and the second-order $(F_2, \rho_2)$ convexity of $-(f(x^1, .) - (.)^T E_1 z^1)$ at $y^1$, we have

$$f(x^1, v^1) + (x_1)^T D_1 w^1 - f(u^1, v^1) - (u_1)^T D_1 w^1 + \frac{1}{2} q^T \nabla_{x^1 x^1} f(u^1, v^1) q$$

$$\ge \ F_1(x^1, u^1; \nabla_{x^1} f(u^1, v^1) + D_1 w^1 + \nabla_{x^1 x^1} f(u^1, v^1) q) + \rho_1 d_1^2(x^1, u^1) \tag{13}$$

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and

$$f(x^1,y^1)-(y^1)^T E_1 z^1 - f(x^1,v^1)+(v^1)^T E_1 z^1 - \frac{1}{2}p^T\nabla_{y^1y^1} f(x^1,y^1)p$$
$$\geq F_2(v^1,y^1;-(\nabla_{y^1} f(x^1,y^1)-E_1z^1+\nabla_{y^1y^1} f(x^1,y^1)p))+\rho_2 d_2^2(v^1,y^1). \qquad (14)$$

Adding the inequalities (13) and (14), we obtain

$$f(x^1,y^1)-f(u^1,v^1)+(x_1)^T D_1 w^1-(u_1)^T D_1 w^1-(y^1)^T E_1 z^1+(v^1)^T E_1 z^1$$
$$+\frac{1}{2}q^T\nabla_{x^1x^1} f(u^1,v^1)q-\frac{1}{2}p^T\nabla_{y^1y^1} f(x^1,y^1)p \geq F_1(x^1,u^1;\nabla_{x^1} f(u^1,v^1)+D_1w^1+\nabla_{x^1x^1} f(u^1,v^1)q)$$
$$+F_2(v^1,y^1;-(\nabla_{y^1} f(x^1,y^1)-E_1z^1+\nabla_{y^1y^1} f(x^1,y^1)p))+\rho_1 d_1^2(x^1,u^1)+\rho_2 d_2^2(v^1,y^1). \qquad (15)$$

Since $(x^1,y^1,x^2,y^2,z^1,z^2,p,r)$ is feasible for primal problem (SMP) and $(u^1,v^1,u^2,v^2,w^1,w^2,q,s)$ is feasible for dual problem (SMD), by the dual constraint (7), the vector $a^1=\nabla_{x^1} f(u^1,v^1)+D_1w^1+\nabla_{x^1x^1} f(u^1,v^1)q \in C_1^*$, and so from the hypothesis $(A)$, we obtain

$$F_1(x^1,u^1;a^1)+(a^1)^T u^1 \geq 0. \qquad (16)$$

Similarly,

$$F_2(v^1,y^1;a^2)+(a^2)^T y^1 \geq 0, \qquad (17)$$

for the vector $a^2=-[\nabla_{y^1} f(x^1,y^1)-E_1z^1+\nabla_{y^1y^1} f(x^1,y^1)p] \in C_3^*$.

Using (16) and (17) and hypothesis $(iii)$ in (15), we have

$$f(x^1,y^1)-f(u^1,v^1)+(x_1)^T D_1 w^1-(u_1)^T D_1 w^1-(y^1)^T E_1 z^1+(v^1)^T E_1 z^1$$
$$+\frac{1}{2}q^T\nabla_{x^1x^1} f(u^1,v^1)q-\frac{1}{2}p^T\nabla_{y^1y^1} f(x^1,y^1)p \geq -(u^1)^T a^1-(y^1)^T a^2.$$

Substituting the values of $a^1$ and $a^2$, we get

$$f(x^1,y^1)+(x^1)^T D_1 w^1-(y^1)^T[\nabla_{y^1} f(x^1,y^1)+\nabla_{y^1y^1} f(x^1,y^1)p]-\frac{1}{2}p^T\nabla_{y^1y^1} f(x^1,y^1)p$$
$$\geq f(u^1,v^1)-(v^1)^T E_1 z^1-(u^1)^T[\nabla_{x^1} f(u^1,v^1)+\nabla_{x^1x^1} f(u^1,v^1)q]-\frac{1}{2}q^T\nabla_{x^1x^1} f(u^1,v^1)q.$$

Applying the Schwartz inequality and using (4) and (10), we have

$$f(x^1,y^1)+((x^1)^T D_1 x^1)^{\frac{1}{2}}-(y^1)^T[\nabla_{y^1} f(x^1,y^1)+\nabla_{y^1y^1} f(x^1,y^1)p]-\frac{1}{2}p^T\nabla_{y^1y^1} f(x^1,y^1)p$$
$$\geq f(u^1,v^1)-((v^1)^T E_1 v^1)^{\frac{1}{2}}-(u^1)^T[\nabla_{x^1} f(u^1,v^1)+\nabla_{x^1x^1} f(u^1,v^1)q]-\frac{1}{2}q^T\nabla_{x^1x^1} f(u^1,v^1)q. \qquad (18)$$

By hypothesis $(C)$ and the dual constraint (8), we obtain

$$G_1(x^2,u^2;\nabla_{x^2} g(u^2,v^2)+D_2w^2+\nabla_{x^2x^2} g(u^2,v^2)s) \geq -(u^2)^T[\nabla_{x^2} g(u^2,v^2)+D_2w^2+\nabla_{x^2x^2} g(u^2,v^2)s],$$

which on using the dual constraint (9) yields

$$G_1(x^2,u^2;\nabla_{x^2} g(u^2,v^2)+D_2w^2+\nabla_{x^2x^2} g(u^2,v^2)s) \geq 0.$$

Since $g(.,v^2)+(.)^T D_2w^2$ is second-order $(G_1,\sigma_1)$ pseudo-convex at $u^2$, we have

$$g(x^2,v^2)+(x^2)^T D_2w^2 \geq g(u^2,v^2)+(u^2)^T D_2w^2 - \frac{1}{2}s^T\nabla_{x^2x^2} g(u^2,v^2)s+\sigma_1 d_3^2(x^2,u^2). \qquad (19)$$

Similarly, from (2) and (3) and hypothesis $(D)$, along with second-order $(G_2,\sigma_2)$ pseudo-convexity of $-(g(x^2,.)-(.)^T E_2z^2)$ at $y^2$, we get

$$g(x^2,y^2)-(y^2)^T E_2z^2 \geq g(x^2,v^2)-(v^2)^T E_2z^2+\frac{1}{2}r^T\nabla_{y^2y^2} g(x^2,y^2)r+\sigma_2 d_4^2(v^2,y^2). \qquad (20)$$

Adding inequalities (19) and (20) and using hypothesis $(iv)$, we obtain

$$g(x^2,y^2)+(x^2)^T D_2w^2-(y^2)^T E_2z^2-\frac{1}{2}r^T\nabla_{y^2y^2} g(x^2,y^2)r$$
$$\geq g(u^2,v^2)+(u^2)^T D_2w^2-(v^2)^T E_2z^2-\frac{1}{2}s^T\nabla_{x^2x^2} g(u^2,v^2)s.$$

Applying the Schwartz inequality and using (5) and (11), we have

$$g(x^2,y^2)+((x^2)^T D_2x^2)^{\frac{1}{2}}-(y^2)^T E_2z^2-\frac{1}{2}r^T\nabla_{y^2y^2} g(x^2,y^2)r$$
$$\geq g(u^2,v^2)+(u^2)^T D_2w^2-((v^2)^T E_2v^2)^{\frac{1}{2}}-\frac{1}{2}s^T\nabla_{x^2x^2} g(u^2,v^2)s. \qquad (21)$$

The expressions (18) and (21) together yield

$$f(x^1,y^1)+((x^1)^T D_1x^1)^{\frac{1}{2}}+g(x^2,y^2)+((x^2)^T D_2x^2)^{\frac{1}{2}}-(y^2)^T E_2z^2-(y^1)^T[\nabla_{y^1} f(x^1,y^1)$$
$$+\nabla_{y^1y^1} f(x^1,y^1)p]-\frac{1}{2}p^T\nabla_{y^1y^1} f(x^1,y^1)p-\frac{1}{2}r^T\nabla_{y^2y^2} g(x^2,y^2)r \geq f(u^1,v^1)-((v^1)^T E_1v^1)^{\frac{1}{2}}$$
$$+g(u^2,v^2)-((v^2)^T E_2v^2)^{\frac{1}{2}}+(u^2)^T D_2w^2-(u^1)^T[\nabla_{x^1} f(u^1,v^1)+\nabla_{x^1x^1} f(u^1,v^1)q]$$
$$-\frac{1}{2}q^T\nabla_{x^1x^1} f(u^1,v^1)q-\frac{1}{2}s^T\nabla_{x^2x^2} g(u^2,v^2)s,$$

that is,

$$L(x^1,y^1,x^2,y^2,z^2,p,r) \geq M(u^1,v^1,u^2,v^2,w^2,q,s).$$

**Theorem 2 (Weak duality)**

Let $(x^1,y^1,x^2,y^2,z^1,z^2,p,r)$ be feasible for SMP and $(u^1,v^1,u^2,v^2,w^1,w^2,q,s)$ be feasible for SMD. Let the sub-linear functionals $F_1: R^{|J_1|}\times R^{|J_1|}\times R^{|J_1|} \mapsto R$, $F_2: R^{|K_1|}\times R^{|K_1|}\times R^{|K_1|} \mapsto R$, $G_1: R^{|J_2|}\times R^{|J_2|}\times R^{|J_2|} \mapsto R$ and $G_2: R^{|K_2|}\times R^{|K_2|}\times R^{|K_2|} \mapsto R$ satisfy the following conditions:

$$F_1(x^1,u^1;a^1)+(a^1)^T u^1 \geq 0, \text{ for all } a^1 \in C_1^*, \qquad (A)$$

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$$F_2(v^1,y^1;a^2)+(a^2)^T y^1 \geq 0, \text{ for all } a^2 \in C_3^*, \qquad (B)$$

$$G_1(x^2,u^2;b^1)+(b^1)^T u^2 \geq 0, \text{ for all } b^1 \in C_2^*, \qquad (C)$$

$$G_2(v^2,y^2;b^2)+(b^2)^T y^2 \geq 0, \text{ for all } b^2 \in C_4^*. \qquad (D)$$

Suppose that

*(i)* $f(.,v^1)+(.)^T D_1 w^1$ is second-order $(F_1,\rho_1)$ convex at $u^1$, and $-(f(x^1,.)-(.)^T E_1 z^1)$ is second-order $(F_2,\rho_2)$ convex at $y^1$,

*(ii)* $g(.,v^2)+(.)^T D_2 w^2$ is second-order $(G_1,\sigma_1)$ convex at $u^2$, and $-(g(x^2,.)-(.)^T E_2 z^2)$ is second-order $(G_2,\sigma_2)$ convex at $y^2$,

*(iii)* either $\rho_1 d_1^2(x^1,u^1)+\rho_2 d_2^2(v^1,y^1) \geq 0$ or $\rho_1,\rho_2 \geq 0$, and

*(iv)* either $\sigma_1 d_3^2(x^2,u^2)+\sigma_2 d_4^2(v^2,y^2) \geq 0$ or $\sigma_1,\sigma_2 \geq 0$.

Then

$L(x^1,y^1,x^2,y^2,z^2,p,r) \geq M(u^1,v^1,u^2,v^2,w^2,q,s)$.

**Proof.** By the second-order $(F_1,\rho_1)$ convexity of $f(.,v^1)+(.)^T D_1 w^1$ at $u^1$ and the second-order $(F_2,\rho_2)$ convexity of $-(f(x^1,.)-(.)^T E_1 z^1)$ at $y^1$, we have

$$f(x^1,v^1)+(x_1)^T D_1 w^1 - f(u^1,v^1)-(u_1)^T D_1 w^1 + \frac{1}{2} q^T \nabla_{x^1x^1} f(u^1,v^1) q$$
$$\geq F_1(x^1,u^1;\nabla_{x^1} f(u^1,v^1)+D_1 w^1 + \nabla_{x^1x^1} f(u^1,v^1)q)+\rho_1 d_1^2(x^1,u^1) \qquad (22)$$

and

$$f(x^1,y^1)-(y^1)^T E_1 z^1 - f(x^1,v^1)+(v^1)^T E_1 z^1 - \frac{1}{2} p^T \nabla_{y^1y^1} f(x^1,y^1) p$$
$$\geq F_2(v^1,y^1;-(\nabla_{y^1} f(x^1,y^1)-E_1 z^1+\nabla_{y^1y^1} f(x^1,y^1)p))+\rho_2 d_2^2(v^1,y^1). \qquad (23)$$

Adding the inequalities (22) and (23), we obtain

$$f(x^1,y^1)-f(u^1,v^1)+(x_1)^T D_1 w^1 -(u_1)^T D_1 w^1 -(y^1)^T E_1 z^1 +(v^1)^T E_1 z^1$$
$$+\frac{1}{2} q^T \nabla_{x^1x^1} f(u^1,v^1) q - \frac{1}{2} p^T \nabla_{y^1y^1} f(x^1,y^1) p \geq F_1(x^1,u^1;\nabla_{x^1} f(u^1,v^1)+D_1 w^1 + \nabla_{x^1x^1} f(u^1,v^1)q)$$
$$+F_2(v^1,y^1;-(\nabla_{y^1} f(x^1,y^1)-E_1 z^1+\nabla_{y^1y^1} f(x^1,y^1)p))+\rho_1 d_1^2(x^1,u^1)+\rho_2 d_2^2(v^1,y^1). \qquad (24)$$

Since $(x^1,y^1,x^2,y^2,z^1,z^2,p,r)$ is feasible for primal problem (SMP) and $(u^1,v^1,u^2,v^2,w^1,w^2,q,s)$ is feasible for dual problem (SMD), by the dual constraint (7), the vector $a^1 = \nabla_{x^1} f(u^1,v^1)+D_1 w^1 + \nabla_{x^1x^1} f(u^1,v^1) q \in C_1^*$, and so from hypothesis $(A)$, we obtain

$$F_1(x^1,u^1;a^1)+(a^1)^T u^1 \geq 0. \qquad (25)$$

Similarly,

$$F_2(v^1,y^1;a^2)+(a^2)^T y^1 \geq 0, \tag{26}$$

for the vector $a^2 = -[\nabla_{y^1} f(x^1,y^1) - E_1 z^1 + \nabla_{y^1 y^1} f(x^1,y^1)p] \in C_3^*$.

Using (25) and (26) and hypothesis *(iii)* in (24), we have

$$f(x^1,y^1) - f(u^1,v^1) + (x_1)^T D_1 w^1 - (u_1)^T D_1 w^1 - (y^1)^T E_1 z^1 + (v^1)^T E_1 z^1$$
$$+\frac{1}{2} q^T \nabla_{x^1 x^1} f(u^1,v^1) q - \frac{1}{2} p^T \nabla_{y^1 y^1} f(x^1,y^1) p \geq -(u^1)^T a^1 - (y^1)^T a^2.$$

Substituting the values of $a^1$ and $a^2$, we get

$$f(x^1,y^1) + (x^1)^T D_1 w^1 - (y^1)^T [\nabla_{y^1} f(x^1,y^1) + \nabla_{y^1 y^1} f(x^1,y^1)p] - \frac{1}{2} p^T \nabla_{y^1 y^1} f(x^1,y^1) p$$
$$\geq f(u^1,v^1) - (v^1)^T E_1 z^1 - (u^1)^T [\nabla_{x^1} f(u^1,v^1) + \nabla_{x^1 x^1} f(u^1,v^1) q] - \frac{1}{2} q^T \nabla_{x^1 x^1} f(u^1,v^1) q.$$

Applying the Schwartz inequality and using (4) and (10), we have

$$f(x^1,y^1) + ((x^1)^T D_1 x^1)^{\frac{1}{2}} - (y^1)^T [\nabla_{y^1} f(x^1,y^1) + \nabla_{y^1 y^1} f(x^1,y^1)p] - \frac{1}{2} p^T \nabla_{y^1 y^1} f(x^1,y^1) p$$
$$\geq f(u^1,v^1) - ((v^1)^T E_1 v^1)^{\frac{1}{2}} - (u^1)^T [\nabla_{x^1} f(u^1,v^1) + \nabla_{x^1 x^1} f(u^1,v^1) q] - \frac{1}{2} q^T \nabla_{x^1 x^1} f(u^1,v^1) q. \tag{27}$$

By second-order $(G_1,\sigma_1)$ convexity of $g(.,v^2)+(.)^T D_2 w^2$ at $u^2$ and the second-order $(G_2,\sigma_2)$ convexity of $-(g(x^2,.)-(.)^T E_2 z^2)$ at $y^2$, we have

$$g(x^2,v^2) + (x^2)^T D_2 w^2 - g(u^2,v^2) - (u^2)^T D_2 w^2 + \frac{1}{2} s^T \nabla_{x^2 x^2} g(u^2,v^2) s$$
$$\geq G_1(x^2,u^2;\nabla_{x^2} g(u^2,v^2) + D_2 w^2 + \nabla_{x^2 x^2} g(u^2,v^2)s) + \sigma_1 d_3^2(x^2,u^2). \tag{28}$$

and

$$g(x^2,y^2) - (y^2)^T E_2 z^2 - g(x^2,v^2) + (v^2)^T E_2 z^2 - \frac{1}{2} r^T \nabla_{y^2 y^2} g(x^2,y^2) r$$
$$\geq G_2(v^2,y^2;-(\nabla_{y^2} g(x^2,y^2) - E_2 z^2 + \nabla_{y^2 y^2} g(x^2,y^2)r)) + \sigma_2 d_4^2(v^2,y^2). \tag{29}$$

Adding the inequalities (28) and (29), we get

$$g(x^2,y^2) + (x^2)^T D_2 w^2 - (y^2)^T E_2 z^2 - g(u^2,v^2) - (u^2)^T D_2 w^2 + (v^2)^T E_2 z^2$$
$$+\frac{1}{2} s^T \nabla_{x^2 x^2} g(u^2,v^2) s - \frac{1}{2} r^T \nabla_{y^2 y^2} g(x^2,y^2) r \geq G_1(x^2,u^2;\nabla_{x^2} g(u^2,v^2) + D_2 w^2 + \nabla_{x^2 x^2} g(u^2,v^2)s)$$
$$+G_2(v^2,y^2;-(\nabla_{y^2} g(x^2,y^2) - E_2 z^2 + \nabla_{y^2 y^2} g(x^2,y^2)r)) + \sigma_1 d_3^2(x^2,u^2) + \sigma_2 d_4^2(v^2,y^2). \tag{30}$$

Since $(x^1,y^1,x^2,y^2,z^1,z^2,p,r)$ is feasible for primal problem (SMP) and $(u^1,v^1,u^2,v^2,w^1,w^2,q,s)$ is feasible for dual problem (SMD), by the dual constraint (8), the vector $b^1 = \nabla_{x^2} g(u^2,v^2) + D_2 w^2 + \nabla_{x^2 x^2} g(u^2,v^2)s \in C_2^*$, and so from hypothesis $(C)$, we obtain

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$$G_1(x^2,u^2;\nabla_{x^2} g(u^2,v^2)+D_2w^2+\nabla_{x^2x^2} g(u^2,v^2)s) \geq -(u^2)^T[\nabla_{x^2} g(u^2,v^2)+D_2w^2+\nabla_{x^2x^2} g(u^2,v^2)s],$$

which on using the dual constraint (9) yields

$$G_1(x^2,u^2;\nabla_{x^2} g(u^2,v^2)+D_2w^2+\nabla_{x^2x^2} g(u^2,v^2)s) \geq 0. \tag{31}$$

Similarly, from (2), (3) and hypothesis $(D)$, we have

$$G_2(v^2,y^2;-(\nabla_{y^2} g(x^2,y^2)-E_2z^2+\nabla_{y^2y^2} g(x^2,y^2)r)) \geq 0. \tag{32}$$

Using (31), (32) and hypothesis $(iv)$ in (30), we obtain

$$g(x^2,y^2)+(x^2)^T D_2w^2-(y^2)^T E_2z^2-\frac{1}{2}r^T\nabla_{y^2y^2} g(x^2,y^2)r$$
$$\geq g(u^2,v^2)+(u^2)^T D_2w^2-(v^2)^T E_2z^2-\frac{1}{2}s^T\nabla_{x^2x^2} g(u^2,v^2)s.$$

Applying the Schwartz inequality and using (5) and (11), we have

$$g(x^2,y^2)+((x^2)^T D_2x^2)^{\frac{1}{2}}-(y^2)^T E_2z^2-\frac{1}{2}r^T\nabla_{y^2y^2} g(x^2,y^2)r$$
$$\geq g(u^2,v^2)+(u^2)^T D_2w^2-((v^2)^T E_2v^2)^{\frac{1}{2}}-\frac{1}{2}s^T\nabla_{x^2x^2} g(u^2,v^2)s. \tag{33}$$

Inequalities (27) and (33) together yield

$$f(x^1,y^1)+((x^1)^T D_1x^1)^{\frac{1}{2}}+g(x^2,y^2)+((x^2)^T D_2x^2)^{\frac{1}{2}}-(y^2)^T E_2z^2-(y^1)^T[\nabla_{y^1} f(x^1,y^1)$$
$$+\nabla_{y^1y^1} f(x^1,y^1)p]-\frac{1}{2}p^T\nabla_{y^1y^1} f(x^1,y^1)p-\frac{1}{2}r^T\nabla_{y^2y^2} g(x^2,y^2)r \geq f(u^1,v^1)-((v^1)^T E_1v^1)^{\frac{1}{2}}$$
$$+g(u^2,v^2)-((v^2)^T E_2v^2)^{\frac{1}{2}}+(u^2)^T D_2w^2-(u^1)^T[\nabla_{x^1} f(u^1,v^1)+\nabla_{x^1x^1} f(u^1,v^1)q]$$
$$-\frac{1}{2}q^T\nabla_{x^1x^1} f(u^1,v^1)q-\frac{1}{2}s^T\nabla_{x^2x^2} g(u^2,v^2)s,$$

that is,

$$L(x^1,y^1,x^2,y^2,z^2,p,r) \geq M(u^1,v^1,u^2,v^2,w^2,q,s).$$

**Theorem 3 (Strong duality)**

Let $f:R^{|J_1|}\times R^{|K_1|}\to R$ and $g:R^{|J_2|}\times R^{|K_2|}\to R$ be differentiable functions and let $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{z}^1,\bar{z}^2,\bar{p},\bar{r})$ be a local optimal solution of SMP. Suppose that

*(i)* the matrix $\nabla_{y^1y^1} f(\bar{x}^1,\bar{y}^1)$ is non-singular,

*(ii)* $\nabla_{y^2y^2} g(\bar{x}^2,\bar{y}^2)$ is positive definite and $\bar{r}^T(\nabla_{y^2} g(\bar{x}^2,\bar{y}^2)-E_2\bar{z}^2)\geq 0$ or $\nabla_{y^2y^2} g(\bar{x}^2,\bar{y}^2)$ is negative definite and $\bar{r}^T(\nabla_{y^2} g(\bar{x}^2,\bar{y}^2)-E_2\bar{z}^2)\leq 0$,

*(iii)* $\nabla_{y^2} g(\bar{x}^2,\bar{y}^2)-E_2\bar{z}^2+\nabla_{y^2y^2} g(\bar{x}^2,\bar{y}^2)\bar{r}\neq 0$, and

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*(iv)* one of the matrices $\frac{\partial}{\partial y_i^1}\left(\nabla_{y^1y^1} f(\bar{x}^1,\bar{y}^1)\right), i=1,2,\ldots,|K_1|,$ is positive or negative definite.

Then $\bar{p}=0,$ $\bar{r}=0$ and there exist $\bar{w}^1 \in R^{|J_1|}$ and $\bar{w}^2 \in R^{|J_2|}$ such that $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{w}^1,\bar{w}^2,\bar{q}=0,\bar{s}=0)$ is feasible for SMD and the objective function values of SMP and SMD are equal. Furthermore, if the assumptions of weak duality theorem (1 or 2) are satisfied for all feasible solutions of SMP and SMD, then $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{z}^1,\bar{z}^2,\bar{p},\bar{r})$ and $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{w}^1,\bar{w}^2,\bar{q},\bar{s})$ are global optimal solutions for SMP and SMD respectively.

**Proof.** Since $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{z}^1,\bar{z}^2,\bar{p},\bar{r})$ is a local solution of SMP, there exist $\alpha \in R_+$, $\beta \in C_3$, $\gamma \in C_4$, $\delta \in R_+$, $\mu \in R_+$ and $\nu \in R_+$ such that the following by Fritz John optimality conditions studied in Suneja et al. [20] and in Schechter [21] are satisfied at $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{z}^1,\bar{z}^2,\bar{p},\bar{r})$:

$$\{\alpha^T(\nabla_{x^1} f(\bar{x}^1,\bar{y}^1)+D_1\bar{w}^1)+\nabla_{y^1x^1} f(\bar{x}^1,\bar{y}^1)\left[\beta-\alpha\bar{y}^1\right]$$
$$+\nabla_{x^1}(\nabla_{y^1y^1} f(\bar{x}^1,\bar{y}^1)\bar{p})[\beta-\alpha(\bar{y}^1+\frac{1}{2}\bar{p})]\}(x^1-\bar{x}^1) \geq 0,\ \forall\, x^1 \in C_1, \tag{34}$$

$$\{\alpha^T(\nabla_{x^2} g(\bar{x}^2,\bar{y}^2)+D_2\bar{w}^2)+\nabla_{y^2x^2} g(\bar{x}^2,\bar{y}^2)\left[\gamma-\delta\bar{y}^2\right]$$
$$+\nabla_{x^2}(\nabla_{y^2y^2} g(\bar{x}^2,\bar{y}^2)\bar{r})[\gamma-\delta\bar{y}^2-\frac{1}{2}\alpha\bar{r}]\}(x^2-\bar{x}^2) \geq 0,\ \forall\, x^2 \in C_2, \tag{35}$$

$$\nabla_{y^1y^1} f(\bar{x}^1,\bar{y}^1)[\beta-\alpha(\bar{y}^1+\bar{p})]+\nabla_{y^1}(\nabla_{y^1y^1} f(\bar{x}^1,\bar{y}^1)\bar{p})[\beta-\alpha(\bar{y}^1+\frac{1}{2}\bar{p})]=0, \tag{36}$$

$$(\nabla_{y^2} g(\bar{x}^2,\bar{y}^2)-E_2\bar{z}^2)\left[\alpha-\delta\right]+\nabla_{y^2y^2} g(\bar{x}^2,\bar{y}^2)[\gamma-\delta(\bar{y}^2+\bar{r})]$$
$$+\nabla_{y^2}(\nabla_{y^2y^2} g(\bar{x}^2,\bar{y}^2)\bar{r})[\gamma-\delta\bar{y}^2-\frac{1}{2}\alpha\bar{r}]=0, \tag{37}$$

$$(-\beta E_1+\mu E_1\bar{z}^1)=0, \tag{38}$$

$$\alpha E_2\bar{y}^2+E_2(\gamma-\delta\bar{y}^2)=2\nu E_2\bar{z}^2, \tag{39}$$

$$\nabla_{y^1y^1} f(\bar{x}^1,\bar{y}^1)[\beta-\alpha(\bar{y}^1+\bar{p})]=0, \tag{40}$$

$$\nabla_{y^2y^2} g(\bar{x}^2,\bar{y}^2)[\gamma-\delta\bar{y}^2-\alpha\bar{r}]=0, \tag{41}$$

$$\beta^T[\nabla_{y^1} f(\bar{x}^1,\bar{y}^1)-E_1\bar{z}^1+\nabla_{y^1y^1} f(\bar{x}^1,\bar{y}^1)\bar{p}]=0, \tag{42}$$

$$\gamma^T[\nabla_{y^2} g(\bar{x}^2,\bar{y}^2)-E_2\bar{z}^2+\nabla_{y^2y^2} g(\bar{x}^2,\bar{y}^2)\bar{r}]=0, \tag{43}$$

$$\delta(\bar{y}^2)^T[\nabla_{y^2} g(\bar{x}^2,\bar{y}^2)-E_2\bar{z}^2+\nabla_{y^2y^2} g(\bar{x}^2,\bar{y}^2)\bar{r}]=0, \tag{44}$$

$$(\bar{x}^1)^T D_1\bar{w}^1=((\bar{x}^1)^T D_1\bar{x}^1)^{\frac{1}{2}}, \tag{45}$$

$$(\bar{x}^2)^T D_2\bar{w}^2=((\bar{x}^2)^T D_2\bar{x}^2)^{\frac{1}{2}}, \tag{46}$$

$$(\bar{w}^1)^T D_1\bar{w}^1 \leq 1, \tag{47}$$

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$$(\bar{w}^2)^T D_2 \bar{w}^2 \le 1, \tag{48}$$

$$\mu((\bar{z}^1)^T E_1 \bar{z}^1 - 1) = 0, \tag{49}$$

$$\nu((\bar{z}^2)^T E_2 \bar{z}^2 - 1) = 0, \tag{50}$$

$$(\alpha, \beta, \gamma, \delta, \mu, \nu) \ne 0. \tag{51}$$

Because of the non-singularity of $\nabla_{y^1 y^1} f(\bar{x}^1, \bar{y}^1)$, (40) yields

$$\beta = \alpha(\bar{y}^1 + \bar{p}). \tag{52}$$

Since $\nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2)$ is positive or negative definite, (41) gives

$$\gamma = \delta\bar{y}^2 + \alpha\bar{r}. \tag{53}$$

Now, we claim that $\alpha > 0$. If possible, let $\alpha = 0$; then (53) gives $\gamma = \delta\bar{y}^2$.

Using (53) in (37), we get

$$(\alpha - \delta)[\nabla_{y^2} g(\bar{x}^2, \bar{y}^2) - E_2 \bar{z}^2 + \nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2)\bar{r}] + \frac{1}{2}\nabla_{y^2}(\nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2)\bar{r})\left[\gamma - \delta\bar{y}^2\right] = 0,$$

which, on using hypothesis (*iii*) and $\gamma = \delta\bar{y}^2$, yields $\alpha = \delta$. As $\alpha = 0$, therefore the equations $\alpha = \delta$ and $\gamma = \delta\bar{y}^2$ give $\delta = 0$ and $\gamma = 0$ respectively. Equation (52) gives $\beta = 0$. Also from equations (38) and (49), we have

$$\mu = \mu((\bar{z}^1)^T E_1 \bar{z}^1) = (\bar{z}^1)^T(\mu E_1 \bar{z}^1) = (\bar{z}^1)^T(E_1 \beta) = 0.$$

From (39) and (50), we get $\nu = 0$. Consequently, $(\alpha, \beta, \gamma, \delta, \mu, \nu) = 0$, contradicting (51). Hence

$$\alpha > 0. \tag{54}$$

Subtracting (44) from (43) yields

$$[\gamma - \delta(\bar{y}^2)]^T[\nabla_{y^2} g(\bar{x}^2, \bar{y}^2) - E_2 \bar{z}^2 + \nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2)\bar{r}] = 0.$$

Using (53) and (54) in above equation, we get

$$\bar{r}^T(\nabla_{y^2} g(\bar{x}^2, \bar{y}^2) - E_2 \bar{z}^2) + \bar{r}^T \nabla_{y^2 y^2} g(\bar{x}^2, \bar{y}^2)\bar{r} = 0, \tag{55}$$

which contradicts hypothesis (*ii*) unless

$$\bar{r} = 0. \tag{56}$$

Equation (53) yields

$$\gamma = \delta\bar{y}^2. \tag{57}$$

Using (56) and (57) in (37), we obtain

$$(\alpha - \delta)(\nabla_{y^2} g(\bar{x}^2, \bar{y}^2) - E_2 \bar{z}^2) = 0,$$

which on using hypothesis (*iii*) and (56) gives

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$$\alpha = \delta. \tag{58}$$

Since $\alpha > 0$, then obviously

$$\delta > 0. \tag{59}$$

Now, using (52) and (54) in (36), we get

$$(\nabla_{y^1}(\nabla_{y^1 y^1} f(\bar{x}^1, \bar{y}^1)\bar{p}))\bar{p} = 0,$$

which by hypothesis $(iv)$ implies

$$\bar{p} = 0. \tag{60}$$

By equations (52) and (60), we have

$$\beta = \alpha \bar{y}^1. \tag{61}$$

From (54) and (61), we obtain

$$\bar{y}^1 = \frac{\beta}{\alpha} \in C_3.$$

Using (58)-(61) in (34), we get

$$(x^1 - \bar{x}^1)^T (\nabla_{x^1} f(\bar{x}^1, \bar{y}^1) + D_1 \bar{w}^1) \geq 0, \text{ for all } x^1 \in C_1. \tag{62}$$

Let $x^1 \in C_1$. Then $x^1 + \bar{x}^1 \in C_1$ as $C_1$ is a closed convex cone, and so (62) implies

$$(x^1)^T (\nabla_{x^1} f(\bar{x}^1, \bar{y}^1) + D_1 \bar{w}^1) \geq 0, \text{ for all } x^1 \in C_1.$$

Therefore,

$$\nabla_{x^1} f(\bar{x}^1, \bar{y}^1) + D_1 \bar{w}^1 \in C_1^*. \tag{63}$$

From (35) and (56)-(59), we have

$$(x^2 - \bar{x}^2)^T (\nabla_{x^2} g(\bar{x}^2, \bar{y}^2) + D_2 \bar{w}^2) \geq 0, \text{ for all } x^2 \in C_2. \tag{64}$$

Let $x^2 \in C_2$. Then $x^2 + \bar{x}^2 \in C_2$ as $C_2$ is a closed convex cone, and so (64) implies

$$(x^2)^T (\nabla_{x^2} g(\bar{x}^2, \bar{y}^2) + D_2 \bar{w}^2) \geq 0, \text{ for all } x^2 \in C_2.$$

Therefore,

$$\nabla_{x^2} g(\bar{x}^2, \bar{y}^2) + D_2 \bar{w}^2 \in C_2^*. \tag{65}$$

Also from (57) and (59), we have

$$\bar{y}^2 = \frac{\gamma}{\delta} \in C_4.$$

Now, letting $x^2 = 0$ and $x^2 = 2\bar{x}^2$ in (64), we get

$$(\bar{x}^2)^T (\nabla_{x^2} g(\bar{x}^2, \bar{y}^2) + D_2 \bar{w}^2) = 0. \tag{66}$$

Thus $(\bar{x}^1, \bar{y}^1, \bar{x}^2, \bar{y}^2, \bar{w}^1, \bar{w}^2, \bar{q}=0, \bar{s}=0)$ satisfies the dual constraints from (7) to (12) and so it is a feasible solution for the dual problem (SMD).

Now let $\frac{2\nu}{\alpha} = a$. Then $a \geq 0$ and from (39) and (57)

$$E_2\bar{y}^2 = aE_2\bar{z}^2, \tag{67}$$

which is the condition for equality in the Schwartz inequality. Therefore

$$(\bar{y}^2)^T E_2\bar{z}^2 = ((\bar{y}^2)^T E_2\bar{y}^2)^{\frac{1}{2}}((\bar{z}^2)^T E_2\bar{z}^2)^{\frac{1}{2}}.$$

In the case of $\nu > 0$, (50) gives $(\bar{z}^2)^T E_2\bar{z}^2 = 1$ and so $(\bar{y}^2)^T E_2\bar{z}^2 = ((\bar{y}^2)^T E_2\bar{y}^2)^{\frac{1}{2}}$. In the case of $\nu = 0$, (67) gives $E_2\bar{y}^2 = 0$ and so $(\bar{y}^2)^T E_2\bar{z}^2 = ((\bar{y}^2)^T E_2\bar{y}^2)^{\frac{1}{2}} = 0$. Thus, in either case,

$$(\bar{y}^2)^T E_2\bar{z}^2 = ((\bar{y}^2)^T E_2\bar{y}^2)^{\frac{1}{2}}. \tag{68}$$

By putting $x^1 = 0$ and $x^1 = 2\bar{x}^1$ in (62), we obtain

$$(\bar{x}^1)^T(\nabla_{x^1} f(\bar{x}^1, \bar{y}^1) + D_1\bar{w}^1) = 0. \tag{69}$$

Also, (45) yields

$$(\bar{x}^1)^T\nabla_{x^1} f(\bar{x}^1, \bar{y}^1) = -(\bar{x}^1)^T D_1\bar{w}^1 = -((\bar{x}^1)^T D_1\bar{x}^1)^{\frac{1}{2}}. \tag{70}$$

From (42), (54), (60) and (61) we get

$$(\bar{y}^1)^T\nabla_{y^1} f(\bar{x}^1, \bar{y}^1) = (\bar{y}^1)^T E_1\bar{z}^1. \tag{71}$$

Equation (38), implies

$$E_1\beta = \mu E_1\bar{z}^1.$$

Using (61) in the above equation,

$$E_1\bar{y}^1 = \frac{\mu}{\alpha}E_1\bar{z}^1. \tag{72}$$

Since equation (72) is the condition for the Schwartz inequality to hold as equality, so

$$(\bar{y}^1)^T E_1\bar{z}^1 = ((\bar{y}^1)^T E_1\bar{y}^1)^{\frac{1}{2}}((\bar{z}^1)^T E_1\bar{z}^1)^{\frac{1}{2}}.$$

In the case of $\mu > 0$, the equation (49) implies $(\bar{z}^1)^T E_1\bar{z}^1 = 1$ and so $(\bar{y}^1)^T E_1\bar{z}^1 = ((\bar{y}^1)^T E_1\bar{y}^1)^{\frac{1}{2}}$.

In the case of $\mu = 0$, the equation (72) gives $E_1\bar{y}^1 = 0$ and so $(\bar{y}^1)^T E_1\bar{z}^1 = ((\bar{y}^1)^T E_1\bar{y}^1)^{\frac{1}{2}} = 0$.

Thus, in either case $(\bar{y}^1)^T E_1\bar{z}^1 = ((\bar{y}^1)^T E_1\bar{y}^1)^{\frac{1}{2}}$.

Now equation (71) becomes

$$(\bar{y}^1)^T\nabla_{y^1} f(\bar{x}^1, \bar{y}^1) = (\bar{y}^1)^T E_1\bar{z}^1 = ((\bar{y}^1)^T E_1\bar{y}^1)^{\frac{1}{2}}. \tag{73}$$

Therefore, using (46), (56), (60), (68), (70) and (73), we obtain the following:

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$$f(\bar{x}^1,\bar{y}^1)+((\bar{x}^1)^T D_1\bar{x}^1)^{\frac{1}{2}}+g(\bar{x}^2,\bar{y}^2)+((\bar{x}^2)^T D_2\bar{x}^2)^{\frac{1}{2}}-(\bar{y}^2)^T E_2\bar{z}^2-(\bar{y}^1)^T[\nabla_{y^1} f(\bar{x}^1,\bar{y}^1)$$
$$+\nabla_{y^1y^1} f(\bar{x}^1,\bar{y}^1)\bar{p}]-\frac{1}{2}\bar{p}^T\nabla_{y^1y^1} f(\bar{x}^1,\bar{y}^1)\bar{p}-\frac{1}{2}\bar{r}^T\nabla_{y^2y^2} g(\bar{x}^2,\bar{y}^2)\bar{r} = f(\bar{x}^1,\bar{y}^1)-((\bar{y}^1)^T E_1\bar{y}^1)^{\frac{1}{2}}$$
$$+g(\bar{x}^2,\bar{y}^2)-((\bar{y}^2)^T E_2\bar{y}^2)^{\frac{1}{2}}+(\bar{x}^2)^T D_2\bar{w}^2-(\bar{x}^1)^T[\nabla_{x^1} f(\bar{x}^1,\bar{y}^1)+\nabla_{x^1x^1} f(\bar{x}^1,\bar{y}^1)\bar{q}]$$
$$-\frac{1}{2}\bar{q}^T\nabla_{x^1x^1} f(\bar{x}^1,\bar{y}^1)\bar{q}-\frac{1}{2}\bar{s}^T\nabla_{x^2x^2} g(\bar{x}^2,\bar{y}^2)\bar{s},$$

that is, the two objective function values are equal.

Finally, from Theorem 1 or 2, we get that $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{z}^1,\bar{z}^2,\bar{p},\bar{r})$ and $(\bar{x}^1,\bar{y}^1,\bar{x}^2,\bar{y}^2,\bar{w}^1,\bar{w}^2,\bar{q},\bar{s})$ are global optimal solutions for SMP and SMD respectively.

**Theorem 4 (Converse duality)**

Let $f: R^{|J_1|}\times R^{|K_1|}\to R$ and $g: R^{|J_2|}\times R^{|K_2|}\to R$ be differentiable functions and let $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{w}^1,\bar{w}^2,\bar{q},\bar{s})$ be a local optimal solution of SMD. Suppose that

*(i)* the matrix $\nabla_{x^1x^1} f(\bar{u}^1,\bar{v}^1)$ is non-singular,

*(ii)* $\nabla_{x^2x^2} g(\bar{u}^2,\bar{v}^2)$ is positive definite and $\bar{s}^T(\nabla_{x^2} g(\bar{u}^2,\bar{v}^2)+D_2\bar{w}^2)\geq 0$ or $\nabla_{x^2x^2} g(\bar{u}^2,\bar{v}^2)$ is negative definite and $\bar{s}^T(\nabla_{x^2} g(\bar{u}^2,\bar{v}^2)+D_2\bar{w}^2)\leq 0$,

*(iii)* $\nabla_{x^2} g(\bar{u}^2,\bar{v}^2)+D_2\bar{w}^2+\nabla_{x^2x^2} g(\bar{u}^2,\bar{v}^2)\bar{s}\neq 0$, and

*(iv)* one of the matrices $\frac{\partial}{\partial x_i^1}\left(\nabla_{x^1x^1} f(\bar{u}^1,\bar{v}^1)\right), i=1,2,\ldots,|J_1|,$ is positive or negative definite.

Then $\bar{q}=0$, $\bar{s}=0$ and there exist $\bar{z}^1\in R^{|K_1|}$ and $\bar{z}^2\in R^{|K_2|}$ such that $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{z}^1,\bar{z}^2,\bar{p}=0,\bar{r}=0)$ is feasible for SMP and the objective function values of SMP and SMD are equal. Furthermore, if the assumptions of weak duality theorem (1 or 2) are satisfied for all feasible solutions of SMP and SMD, then $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{w}^1,\bar{w}^2,\bar{q},\bar{s})$ and $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2,\bar{z}^1,\bar{z}^2,\bar{p},\bar{r})$ are global optimal solutions for SMD and SMP respectively.

**Proof.** It follows on the lines of Theorem 3.

**Special Cases**

In this section, we consider some of the special cases of our problems: SMP and SMD. For all these cases, $D_1=\{0\}$, $D_2=\{0\}$, $E_1=\{0\}$, $E_2=\{0\}$, $C_1=R_+^{|J_1|}, C_2=R_+^{|J_2|}, C_3=R_+^{|K_1|}$ and $C_4=R_+^{|K_2|}$.

1. If $J_2=\emptyset$ and $K_2=\emptyset$, then our problems: SMP and SMD reduce to the programmes: SP and SD studied by Gulati et al. [22] and if $J_1=\emptyset$ and $K_1=\emptyset$ in SMP and SMD, then the programmes SP1 and SD1 in Gulati et al. [22] are obtained.
2. By eliminating the second-order terms, our problems: SMP and SMD reduce to the mixed symmetric dual programmes studied by Chandra et al. [23].

3. If $J_1 = \varnothing$ and $K_1 = \varnothing$, then SMP and SMD reduce to programmes studied by Yang [24] with the omission of non-negativity constraints from SMP and SMD.
4. If $p = 0$, $q = 0$, $J_2 = \varnothing$ and $K_2 = \varnothing$ in SMP and SMD, then the programmes WP and WD in Chandra et al. [25] are obtained and if we take $r = 0$, $s = 0$, $J_1 = \varnothing$ and $K_1 = \varnothing$, then our problems become the programmes MP and MD studied by Chandra et al. [25].

## CONCLUSIONS

Weak, strong and converse duality theorems have been established for a pair of non-differentiable second-order mixed symmetric dual programmes with cone constraints under second-order $(F, \rho)$ convexity/pseudo-convexity assumptions. It is to be noted that previously known results are special cases of our study. However, it is not clear whether the second-order mixed symmetric duality in mathematical programming can be further extended to second-order multi-objective symmetric dual programmes.

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