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*Full Paper*

# p-Absolutely summable sequences of fuzzy real numbers

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**Abstract:** In this paper the fuzzy sequence space $\left(\ell_p\right)_\lambda^F$ is introduced and some algebraic properties such as solidness, symmetricalness, convergence free and sequence algebra are studied, and some inclusion relations for the space $\left(\ell_p\right)_\lambda^F$ are provided.

**Keywords:** p-absolutely summable sequences, fuzzy real numbers, convergence free, sequence algebra

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## INTRODUCTION

The concept of fuzzy sets was first introduced by Zadeh [1]. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [2], who showed that every convergent sequence of fuzzy numbers is bounded. Later on sequences of fuzzy numbers were discussed by Nanda [3], Esi [4], Kaleva and Seikkala [5], Tripathy and Baruah [6-8], Tripathy and Borgogain [9,10], Tripathy and Dutta [11,12], Tripathy and Sarma [13,14], and Tripathy et al. [15]. Briefly, we recall some of the basic notations in the theory of fuzzy numbers and for more information one may refer to Matloka [2] and Diamond and Kloeden [16] for more details.

A fuzzy number $X$ is a fuzzy subset of the real line $\mathrm{R}$, i.e. a mapping $X:\mathrm{R}\to J(=[0,1])$ associating each real number $t$ with its grade of membership $X(t)$. A fuzzy number $X$ is *convex* if $X(t)\geq X(s)\wedge X(r)=min\{X(s),X(r)\}$, where $s<t<r$. If there exists $t_0\in\mathrm{R}$ such that $X(t_0)=1$, then the fuzzy number $X$ is called *normal*. A fuzzy number $X$ is said to be *upper-semi continuous* if for each $\varepsilon > 0$ and for all $a\in J$, $X^{-1}([0,a+\varepsilon))$ is open in the usual topology of $\mathrm{R}$. Let $\mathrm{R}(J)$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support, i.e. if $X\in\mathrm{R}(J)$, then for any $\alpha\in[0,1]$, $[X]^\alpha$ is compact, where

$$[X]^\alpha=\{t\in\mathrm{R}:X(t)\geq\alpha,\ \text{if } \alpha\in[0,1]\},\ [X]^0=closure\ \ of\ \ (\{t\in\mathrm{R}:X(t)>\alpha,\ \text{if } \alpha=0\}).$$

The set $\mathrm{R}$ of real numbers can be embedded in $\mathrm{R}(J)$ if we define $\bar{r}\in\mathrm{R}(J)$ by

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$$\bar{r}(t)=\begin{cases}1, \text{if} t=r\\ 0, \text{if} t\neq r\end{cases}.$$

The additive identity and multiplicative identity of $\mathbb{R}(J)$ are defined by $\bar{0}$ and $\bar{1}$ respectively.
The arithmetic operations on $\mathbb{R}(J)$ are defined as follows:

$$(X\oplus Y)(t)=\sup\{X(s)\wedge Y(t-s)\}, t\in\mathbb{R},$$

$$(X\,\Theta\,Y)(t)=\sup\{X(s)\wedge Y(s-t)\}, t\in\mathbb{R},$$

$$(X\otimes Y)(t)=\sup\{X(s)\wedge Y(\frac{t}{s})\}, t\in\mathbb{R},$$

$$(\frac{X}{Y})(t)=\sup\{X(st)\wedge Y(s)\}, t\in\mathbb{R}, \text{provided } 0\notin[Y]^{0}.$$

Let $X, Y\in\mathbb{R}(J)$ and the $\alpha-$ level sets be $[X]^{\alpha}=[x_1^{\alpha}, x_2^{\alpha}], [Y]^{\alpha}=[y_1^{\alpha}, y_2^{\alpha}], \alpha\in[0,1]$. Then the above operations can be defined in terms of $\alpha-$level sets as follows:

$$[X\oplus Y]^{\alpha}=[x_1^{\alpha}+y_1^{\alpha}, x_2^{\alpha}+y_2^{\alpha}],$$

$$[X\,\Theta\,Y]^{\alpha}=[x_1^{\alpha}-y_1^{\alpha}, x_2^{\alpha}-y_2^{\alpha}],$$

$$[X\otimes Y]^{\alpha}=[\min_{i,j\in\{1,2\}} x_i^{\alpha}y_j^{\alpha}, \max_{i,j\in\{1,2\}} x_i^{\alpha}y_j^{\alpha}],$$

$$[X^{-1}]^{\alpha}=[(x_2^{\alpha})^{-1}, (x_1^{\alpha})^{-1}], x_1^{\alpha}>0, \textit{for each } 0<\alpha\leq 1.$$

For $r\in R$ and $X\in\mathbb{R}(J)$, the product $rX$ is defined as follows:

$$rX(t)=\begin{cases}X(r^{-1}t), \text{if} r\neq 0\\ 0, \text{if} r=0\end{cases}.$$

The absolute value , $|X|$, of $X\in\mathbb{R}(J)$ is defined [5] by

$$|X|(t)=\begin{cases}\max\{X(t), X(-t)\}, \text{if} t\geq 0\\ 0, \text{if} t<0\end{cases}.$$

A mapping $\bar{d}:\mathbb{R}(J)\times\mathbb{R}(J)\to\mathbb{R}^{+}\cup\{0\}$ is defined by

$$\bar{d}(X,Y)=\sup_{0\leq\alpha} d([X]^{\alpha},[Y]^{\alpha}).$$

It is known that $(\mathbb{R}(J),\bar{d})$ is a complete metric space [5].

A metric on $\mathbb{R}(J)$ is said to be *translation invariant* if

$$\bar{d}(X+Z, Y+Z)=\bar{d}(X,Y), \textit{for } X, Y, Z\in R(J).$$

A sequence $X=(X_k)$ of fuzzy numbers is a function $X$ from the set $N$ of natural numbers in $L(R)$. The fuzzy number $X_k$ denotes the value of the function at $k\in\mathbb{N}$ [2].
Let $E^{F}$ denote the sequence space of fuzzy numbers.

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A sequence space $E^F$ is said to be *solid* (or *normal*) if $(Y_k) \in E^F$ whenever $(X_k) \in E^F$ and $|Y_k| \leq |X_k|$ for all $k \in \mathbb{N}$.

A sequence space $E^F$ is said to be *symmetric* if $(X_k) \in E^F$ implies $(X_{\pi(k)}) \in E^F$ where $\pi$ is a permutation of $\mathbb{N}$.

A sequence space $E^F$ is said to be *sequence algebra* if $(X_k \otimes Y_k) \in E^F$ whenever $(X_k),(Y_k) \in E^F$.

A sequence space $E^F$ is said to be *convergence free* if $(Y_k) \in E^F$ whenever $(X_k) \in E^F$ and $X_k = \bar{0}$ implies $Y_k = \bar{0}$.

A sequence space $E^F$ is said to be *monotone* if $E^F$ contains the canonical pre-images of all its step spaces.

**Lemma.** *If a sequence space $E^F$ is normal then it is monotone*. (For the crisp set case, one may refer to Kamthan and Gupta [17]).

In this paper, we define the p-absolutely $\lambda-$ summable sequence space of fuzzy real numbers $(\ell_p)_\lambda^F$ as follows: Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$, $I_n = [n - \lambda_n + 1, n]$ and we define:

$$(\ell_p)_\lambda^F = \left\{ X = (X_k) : \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \bar{d}(X_k, \bar{0}) \right)^p < \infty \right\}$$

where $1 \leq p < \infty$. It is noted that $(\ell_p)_\lambda^F = Ces(p)$ for $\lambda_n = n$ for all $n \in \mathbb{N}$.

## MAIN RESULTS

**Theorem 1.** The sequence space $(\ell_p)_\lambda^F$ is closed under addition and scalar multiplication.

**Proof:** Let X=(X$_k$)$\in (\ell_p)_\lambda^F$ and $\alpha \in \mathbb{R}$. Then

$$\sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \bar{d}(X_k, \bar{0}) \right)^p < \infty.$$

Then we write:

$$\sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \bar{d}(\alpha X_k, \bar{0}) \right)^p = \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{k \in I_n} |\alpha| \bar{d}(X_k, \bar{0}) \right)^p$$

$$= |\alpha|^p \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \bar{d}(X_k, \bar{0}) \right)^p < \infty.$$

This implies that $\alpha$X=($\alpha$X$_k$)$\in (\ell_p)_\lambda^F$. Now let X=(X$_k$), Y=(Y$_k$)$\in (\ell_p)_\lambda^F$. Then

$$\sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \bar{d}(X_k, \bar{0}) \right)^p < \infty \text{ and } \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \sum_{k \in I_n} \bar{d}(Y_k, \bar{0}) \right)^p < \infty.$$

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Then we can write:

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k+Y_k,\bar{0}\right)\right)^p \le \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k,\bar{0}\right)\right)^p + \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(Y_k,\bar{0}\right)\right)^p < \infty.$$

Thus X+Y=($X_k$+$Y_k$) ∈ $(\ell_p)_\lambda^F$.

**Theorem 2.** The sequence space $(\ell_p)_\lambda^F$ is solid and hence monotone.

**Proof:** Let $X=(X_k)$ and $Y=(Y_k)$ be two sequences of fuzzy real numbers such that $\bar{d}(Y_k,\bar{0})\le \bar{d}(X_k,\bar{0})$ for all $k\in$ N. If X=($X_k$) ∈ $(\ell_p)_\lambda^F$. Then

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k,\bar{0}\right)\right)^p < \infty.$$

Now, we have

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(Y_k,\bar{0}\right)\right)^p \le \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k,\bar{0}\right)\right)^p < \infty.$$

Hence Y=($Y_k$) ∈ $(\ell_p)_\lambda^F$. Thus, the sequence space $(\ell_p)_\lambda^F$ is solid and hence monotone.

**Theorem 3.** The sequence space $(\ell_p)_\lambda^F$ is sequence algebra.

**Proof:** Let X=($X_k$), Y=($Y_k$) ∈ $(\ell_p)_\lambda^F$. Then

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k,\bar{0}\right)\right)^p < \infty \text{and} \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(Y_k,\bar{0}\right)\right)^p < \infty.$$

Thus, we can write:

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k\otimes Y_k,\bar{0}\right)\right)^p \le \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k,\bar{0}\right).\bar{d}\left(Y_k,\bar{0}\right)\right)^p$$

$$\le \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k,\bar{0}\right)\right)^p .\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(Y_k,\bar{0}\right)\right)^p < \infty.$$

Thus, ($X_k$⊗$Y_k$) ∈ $(\ell_p)_\lambda^F$. So the sequence space $(\ell_p)_\lambda^F$ is sequence algebra.

**Theorem 4.** The sequence space $(\ell_p)_\lambda^F$ is not symmetric in general.

**Proof:** We shall prove it by the following example:

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**Example 1.** Let $p=1$, $\lambda_n = n$ for all $n \in \mathbb{N}$. Consider the sequence $X=(X_k)$ defined by:

$$X_k(t)=\begin{cases} k^2t+1, \text{if } -k^{-2} \le t \le 0 \\ 1-k^2t, \text{if } 0 \le t \le k^{-2} \\ 0, \text{otherwise} \end{cases}.$$

Then

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k,\bar{0}\right)\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k\in I_n}k^{-2}\right) < \infty.$$

Hence X=(X$_k$)$\in$ $\left(\ell_p\right)_\lambda^F$ . Now we consider the rearrangement of $X=(X_k)$ defined as $Y=(Y_k)=\left(X_1,\bar{0},X_2,\bar{0},X_3,\bar{0},...\right)$ Then we have:

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(Y_k,\bar{0}\right)\right)^p \to \infty.$$

Then Y=(Y$_k$)$\notin \left(\ell_p\right)_\lambda^F$. Hence the sequence space $\left(\ell_p\right)_\lambda^F$ is not symmetric in general.

**Theorem 5.** The sequence space $\left(\ell_p\right)_\lambda^F$ is not convergence free in general.

**Proof:** We shall prove it by the following example:

**Example 2.** Let $p=1$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Consider the sequence $X=(X_k)$ defined by:

$$X_k(t)=\begin{cases} k^2t+1, \text{if } -k^{-2} \le t \le 0 \\ 1-k^2t, \text{if } 0 \le t \le k^{-2} \\ 0, \text{otherwise} \end{cases}.$$

Then

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k,\bar{0}\right)\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k\in I_n}k^{-2}\right) < \infty.$$

Thus, $X=(X_k)\in\left(l_p\right)_\lambda^F$. Now we consider the sequence $Y=\left(Y_k\right)$ defined by:

$$Y_k(t)=\begin{cases} k^{\frac{1}{2}}t+1, \text{if } -k^{-\frac{1}{2}} \le t \le 0 \\ 1-k^{\frac{1}{2}}t, \text{if } 0 \le t \le k^{\frac{-1}{2}} \\ 0, \text{otherwise} \end{cases}.$$

Then

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(Y_k,\bar{0}\right)\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k\in I_n}k^{-\frac{1}{2}}\right) = \infty.$$

Thus, Y=(Y$_k$)$\notin \left(\ell_p\right)_\lambda^F$. Hence the sequence space $\left(\ell_p\right)_\lambda^F$ is not convergence free in general.

**Theorem 6.** Let $0<p<q$. Then $\left(\ell_p\right)_\lambda^F \subset \left(\ell_q\right)_\lambda^F$.

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**Proof:** It is clear from the following inclusion relation: for any $X=(X_k)\in\left(\ell_p\right)_\lambda^F$ ,

$$\sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k,\bar{0}\right)\right)^p \le \sum_{n=1}^{\infty}\left(\frac{1}{\lambda_n}\sum_{k\in I_n}\bar{d}\left(X_k,\bar{0}\right)\right)^q .$$

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