Reachability analysis of a class of Petri nets using place invariants and siphons

Xiu Yan Zhang ¹, Zhi Wu Li ¹,*, Chun Fu Zhong ¹ and Abdulrahman M. Al-Ahmari ²

¹ School of Electro-Mechanical Engineering, Xidian University, No.2 South Taibai Road, Xi’an 710071, China
² Department of Industrial Engineering, College of Engineering, King Saud University, P. O. Box 800, Riyadh 11421, Saudi Arabia
* Corresponding author, e-mail: systemscontrol@gmail.com

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Abstract: This paper proposes a novel and computationally efficient approach to deal with the reachability problem by using place invariants and strict minimal siphons for a class of Petri nets called pipe-line nets (PLNs). First, in a PLN with an appropriate initial marking, the set of invariant markings and the set of strict minimal siphons are enumerated. Then a sufficient and necessary condition is developed to decide whether a marking is spurious by analysing the number of tokens in operation places of any strict minimal siphon and their bounds. Furthermore, an algorithm that generates the reachable markings by removing all the spurious markings from the set of invariant markings is proposed. Finally, experimental results show the efficiency of the proposed method.

Keywords: Petri nets, strict minimal siphons, place invariants, reachability analysis, flexible manufacturing system

INTRODUCTION

A flexible manufacturing system (FMS) is an automatically running system that consists of resources such as machines, robots, buffers, and conveyors. In an FMS, part processing sequences are executed concurrently, which have to compete for the limited system resources. This competition can cause deadlocks when some processes keep waiting indefinitely for other processes to release resources [1]. Deadlocks must be considered in FMSs since they may offset the advantages of these systems and even lead to catastrophic results such as long downtime and low use of some critical and expensive resources. Therefore, it is necessary to ensure that deadlocks will never occur in such a system.
To deal with deadlock problems in FMSs, Petri nets [2-6], automata [7-8], and graph theory [1] are major mathematical tools. Many researchers use Petri nets as a formalism to deal with deadlock problems [9-14]. There are mainly three approaches: deadlock detection and recovery [15-16], deadlock avoidance [17-19] and deadlock prevention [1, 9, 20, 21].

For Petri nets, there are two widely used analysis techniques for deadlock prevention in FMSs: structure analysis [4, 10, 11, 15, 22, 23] and reachability graph analysis [24-27]. The former always derives a deadlock prevention policy by structural objects of Petri nets, such as siphons and resource-transition circuits. The policy is often simple but always restricts the behaviour of a system in the sense that a part of permissive behaviour is excluded. Therefore, it is suboptimal in general. The latter, the reachability graph analysis, can obtain a liveness-enforcing supervisor with highly permissive or even maximally permissive behaviour. However, its computation is always expensive, which always suffers from a state explosion problem since it requires an enumeration of all or a part of reachable markings. Thus, to tackle this problem, it is necessary to explore more efficient approaches to compute reachable markings.

This paper proposes a novel approach to compute the set of reachable markings using P-invariants and strict minimal siphons in a class of Petri nets called PLNs. First, the set of invariant markings and the set of strict minimal siphons of a PLN are enumerated. As known, the set of invariant markings include spurious markings. Then a sufficient and necessary condition to identify the spurious markings is established. Finally the reachability set of the net is generated by removing all the spurious markings from the set of invariant markings.

**PRELIMINARIES**

**Basics of Petri nets**

A Petri net [2] is a four-tuple \( N = (P, T, F, W) \) where \( P \) and \( T \) are finite and non-empty sets. \( P \) is a set of places and \( T \) is a set of transitions with \( P \cap T = \emptyset \). \( F \subseteq (P \times T) \cup (T \times P) \) is called the flow relation of the net, represented by arcs with arrows from places to transitions or from transitions to places. \( W : (P \times T) \cup (T \times P) \rightarrow \text{IN} \) is a mapping that assigns a weight to an arc: \( W(x, y) > 0 \) if \( (x, y) \in F \), and \( W(x, y) = 0 \) otherwise, where \( x, y \in P \cup T \) and \( \text{IN} = \{0,1,2,...\} \) is the set of non-negative integers. \( N = (P, T, F, W) \) is said to be ordinary if \( \forall (x, y) \in F, \ W(x, y) = 1 \). A net \( N = (P, T, F, W) \) is pure (self-loop free) if \( \forall x, y \in P \cup T, \ W(x, y) > 0 \) implies \( W(y, x) = 0 \). A pure net \( N = (P, T, F, W) \) can be represented by its incidence matrix \( [N] \), where \( [N] \) is a \( |P| \times |T| \) integer matrix with \( [N](p, t) = W(t, p) - W(p, t) \).

A marking \( M \) of a Petri net \( N \) is a mapping from \( P \) to \( \text{IN} \). \( M(p) \) denotes the number of tokens in place \( p \). A place \( p \) is marked at \( M \) if \( M(p) > 0 \). A subset \( S \subseteq P \) is marked at \( M \) if at least one place in \( S \) is marked at \( M \). The sum of tokens in all places in \( S \) is denoted by \( M(S) \), i.e., \( M(S) = \sum_{p \in S} M(p) \). \( S \) is said to be empty or unmarked at \( M \) if \( M(S) = 0 \). \( M_0 \) is called an initial marking of \( N \) and \( (N, M_0) \) is called a net system or marked net.

Let \( x \in P \cup T \) be a node of net \( N \). The preset of node \( x \) is defined as \( x^* = \{ y \in P \cup T | (y, x) \in F \} \), while the postset of \( x \) is defined as \( x^+ = \{ y \in P \cup T | (x, y) \in F \} \). These notations can be extended to a set of nodes as follows: given \( X \subseteq P \cup T \), \( x^* = \bigcup_{x \in X} x^* \) and \( X^* = \bigcup_{x \in X} x^* \). For \( t \in T \), \( p \in t^* \) is called an input place of \( t \) and \( p \in t^* \) is called an output place of
A marked graph is an ordinary Petri net satisfying \(|s| = |t'| = 1\), \(\forall t \in T\). A marked graph is an ordinary Petri net satisfying \(|s| = |t'| = 1\), \(\forall p \in P\). A sequence of nodes \(x_1, \ldots, x_n\) is called a path of \(N\) if \(\forall i \in \{1, 2, \ldots, n-1\}\), \(x_i \subseteq x_{i+1}\), where \(x_i \in P \cup T\). An elementary path from \(x_i\) to \(x_n\) is a path whose nodes are all different (perhaps, except for \(x_i\) and \(x_n\)). It is called a circuit if \(x_i = x_n\). A Petri net \(N\) is said to be strongly connected if there is a sequence of nodes \(x, a, b, \ldots, c, y\) in \(N\) for \(\forall x, y \in P \cup T\) such that \((x, a), (a, b), \ldots, (c, y) \in E\), where \(\{a, b, \ldots, c\} \subseteq P \cup T\).

A transition \(t \in T\) is enabled at \(M\) if \(\forall p \in S, M(p) \geq W(p, t)\). This fact is denoted as \(M(t)\). Firing it yields a new marking \(M'\) such that \(\forall p \in P, M'(p) = M(p) - W(p, t) + W(t, p)\), denoted as \(M(t)M'\). \(M'\) is called an immediately reachable marking from \(M\). The marking \(M^*\) is said to be reachable from the marking \(M\) if there exists a sequence of enabled transitions \(\sigma = t_1, \ldots, t_n\) and markings \(M_1, M_2, \ldots, M_n\) such that \(M(t_0)M(t_1)M_2 \ldots M_n M^*\) holds, which is denoted as \(M[\sigma]M^*\). The set of markings reachable from \(M_0\) by firing any possible sequence of transitions in \(N\) is called the reachability set of Petri net \((N, M_0)\) and is denoted by \(R(N, M_0)\). A reachability graph is a directed graph whose nodes are markings in \(R(N, M_0)\) and arcs are labelled by the transitions of \(N\). An arc from \(M_1\) to \(M_2\) is labelled by \(t\) if \(M_1(t)M_2\). \(N'\) is the reverse net of \(N\) obtained by reversing the direction of all arcs in \(N\) with the initial marking unchanged.

A P-vector is a column vector \(I: P \rightarrow Z\) indexed by \(P\) and a T-vector is a column vector \(J: T \rightarrow Z\) indexed by \(T\), where \(Z\) is the set of integers. P-vector \(I\) is called a P-invariant (place invariant) if \(I \neq 0\) and \(I^T[N] = 0^T\). T-vector \(J\) is called a T-invariant (transition invariant) if \(J \neq 0\) and \([N]J = 0\). A P-invariant \(I\) is said to be a P-semiflow if every element of \(I\) is non-negative. \(I\) is called a minimal P-invariant if \(I\) is not a superset of the support of any other one and its components are mutually prime. Let \(I\) be a P-invariant of \((N, M_0)\) and \(M\) be a reachable marking from \(M_0\). Then \(I^T M = I^T M_0\).

Let \(X\) be a matrix where each column is a P-semiflow of the net \((N, M_0)\) and \(I_X(N, M_0) = \{M \in IN^P \mid X^T M = X^T M_0\}\) denotes the set of invariant markings, where \(IN^P\) is a set of non-negative vectors, each of which has a length of \(|P|\). It can be noted that \(R(N, M_0) \subseteq I_X(N, M_0)\).

A non-empty set \(S \subseteq P\) is a siphon if \(*S \subseteq S^*\). \(S \subseteq P\) is a trap if \(S^* \subseteq *S\). A siphon is minimal if there is no siphon contained in it as a proper subset. A minimal siphon is said to be strict if \(*S \subset S^*\).

**S3PR Nets**

**Definition 1** [9]. A simple sequential process (S3P) is a Petri net \(N = (P_A \cup \{p^0\}, T, F)\), where the following statements are true: (1) \(P_A \neq \emptyset\) is a set of operation places; (2) \(p^0 \in P_A\) is called the process idle place; (3) \(N\) is a strongly connected state machine; and (4) every circuit of \(N\) contains place \(p^0\).
Let $N = (P, T, F)$ be an S$^3$P with idle process place $p^0$. Let $C$ be a circuit of $N$, and $x$ and $y$ be two nodes of $C$. Node $x$ is said to be previous to $y$ if there exists a path in $C$ from $x$ to $y$, the length of which is greater than one and does not pass over the idle place $p^0$. This fact is denoted by $x <_c y$. Let $x$ and $y$ be two nodes in $N$. Node $x$ is said to be previous to $y$ in $N$ if there exists a circuit $C$ such that $x <_N y$. This fact is denoted by $x <_N y$.

**Definition 2** [9]. A system of simple sequential processes with resources (S$^3$PR) $N = (P^0 \cup P_A \cup P_R, T, F)$ is defined as the union of a set of nets $N_i = (\{p^0_i\} \cup P_A \cup P_R, T_i, F_i)$ sharing common places, where the following statements are true:

1. $p^0_i$ is called the process idle places of $N_i$. Elements in $P_A$ and $P_R$ are called operation places and resource places respectively;
2. $P_R \neq \phi$; $P_A \neq \phi$; $p^0_i \notin P_A$; $(P_A \cup \{p^0_i\}) \cap P_R = \phi$;
3. $\forall p \in P_A, \forall t \in \ast p, \forall t' \in p^*, \exists r_p \in P_R, t \cap P_R = t' \cap P_R = \{r_p\}$;
4. $\forall r \in P_R, 'r \cap P_R = 'r' \cap P_R = \phi$ and $'r \cap 'r' = \phi$;
5. $\{p^0_i\} \cap P_R = \{p^0_i\}' \cap P_R = \phi$;
6. $N_i'$ is a strongly connected state machine, where $N_i' = (P_A \cup \{p^0_i\}, T_i, F_i)$ is the resulting net after the places in $P_R$ and related arcs are removed from $N_i$. Every circuit of $N_i'$ contains place $p^0_i$;
7. any two $N_i$’s are composable when they share a set of common places. Every shared place must be a resource place; and
8. transitions in $(p^0_i)'$ and $(p^0_i)'$ are called source and sink transitions of an S$^3$PR respectively.

In an S$^3$PR, $P^0$ is called the set of process idle places, $P_A$ is called the set of operation places and $P_R$ is called the set of resource places.

**LS$^3$PR Nets**

**Definition 3** [28]. An S$^3$PR $N = (P, T, F)$ is called a linear S$^3$PR (LS$^3$PR) if

1. $P = P^0 \cup P_A \cup P_R$ is a partition of places, where
   1.a. $P^0 = \{p^0_1, p^0_2, \ldots, p^0_k\}$, $k > 0$,
   1.b. $P_A = \bigcup_{i=1}^k P_{A_i}$, where $P_{A_i} \cap P_{A_j} = \phi$, for all $i \neq j$,
   1.c. $P_R = \{r_1, r_2, \ldots, r_n\}$, $n > 0$;
2. $T = \bigcup_{i=1}^k T_{A_i}$, where $T_{A_i} \cap T_{A_j} = \phi$, for all $i \neq j$;
3. $\forall i \in \{1, 2, \ldots, k\}$, the subnet $N_i$ generated by $P^0_i \cup P_A \cup T_i$ is a strongly connected state machine such that every cycle of $N_i$ contains place $p^0_i$ and $\forall p \in P_{A_i}, |p^*| = 1$;
4. $\forall i \in \{1, 2, \ldots, k\}$, $\forall p \in P_A, ''p \cap P_R = ''p' \cap P_R$ and $''p \cap P_R = 1$ and;
5. $N$ is strongly connected.

**Definition 4** [28]. Let $N = (P^0 \cup P_A \cup P_R, T, F)$ be an LS$^3$PR. Given $p \in P_A$, if $''p \cap P_R = \{r_p\}$, $r_p$ is called the resource used by $p$. For $r \in P_R$, $H(r) = ''r \cap P_A$ is called the set of holders of $r$.

**Definition 5** [28]. Let $N = (P^0 \cup P_A \cup P_R, T, F)$ be an LS$^3$PR. An initial marking $M_0$ is called an admissible initial marking for $N$ if

1. $M_0(p^0) \geq 1, \forall p^0 \in P^0$;
2. $M_0(p) = 0, \forall p \in P_A$; and
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Definition of a PLN

Definition 6. Let $N = (P^0 \cup P_A \cup P_R, T, F)$ be an LS^3PR and $N_i$ be the subnet generated by \{$p_i^0\} \cup P_A \cup T_i$. The two subnets $N_j$ and $N_j$ are said to be mutually reversed if $\exists r, r_i \in P_R (r \neq r_i)$ such that one of the two following statements holds: 1) $p_i \leq_{N_i} p$ and $q_i \leq_{N_i} q_i$; and 2) $p \leq_{N_i} p_i$ and $q_i \leq_{N_i} q_i$.

The net shown in Figure 1 is an LS^3PR with $P_0 = \{p_1, p_2, p_1\}$, $P_A = \{p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9, p_{10}, p_{11}, p_{12}, p_{13}\}$ and $P_R = \{p_{14}, p_{15}, p_{16}, p_{17}, p_{18}\}$. It consists of three subnets: $N_1$ generated by \{$p_2\} \cup \{p_2, p_3, p_4, p_5, p_6\} \cup \{t_1, t_2, t_3, t_4, t_5, t_6\}$, $N_2$ generated by \{$p_7\} \cup \{p_7, p_8, p_9, p_{10}\} \cup \{t_7, t_8, t_9, t_{10}\}$ and $N_3$ generated by \{$p_{13}\} \cup \{p_{12}, p_{13}\} \cup \{t_{11}, t_{12}, t_{13}\}$. In the net, $p_2, p_3, p_4, p_5, p_6 \in P_A$, $p_7, p_8, p_9, p_{10} \in P_A$, $p_{12}, p_{13} \in P_A$, $p_2, p_8 \in H(p_{14})$, $p_3, p_9 \in H(p_{15})$, $p_4, p_{10} \in H(p_{16})$, $p_5, p_{12} \in H(p_{17})$ and $p_6, p_{13} \in H(p_{18})$. Since $p_2 \leq_{N_1} p_3$ and $p_6 
 \leq_{N_1} p_6$; $p_3 \leq_{N_1} p_4$ and $p_{10} \leq_{N_2} p_4$; and $p_5 \leq_{N_1} p_4$ and $p_{10} \leq_{N_1} p_4$, the two subnets $N_1$ and $N_2$ are mutually reversed. Since $p_5 \leq_{N_1} p_6$ and $p_{12} \leq_{N_3} p_{13}$, the two subnets $N_1$ and $N_3$ are not mutually reversed.

[Diagram of PLN]

Definition 7. An LS^3PR $N = (P^0 \cup P_A \cup P_R, T, F)$ is called a PLN if

1. $\forall r \in P_R$, $\forall i \in \{1, 2, \ldots, k\}$, $|H(r)| = 2$ and $|H(r) \cap P_A| \leq 1$; and

2. $q_i \in H^* \cap P_{A_i}$ holds if the two subnets $N_i$ and $N_j$ are mutually reversed with $p, p_i \in P_A$, $q, q_i \in P_{A_i}$, $p_i \in H^* \cap P_A$, $p, q \in H(r)$, $p_i, q_i \in H(r_i)$, $r, r_i \in P_R$ and $r \neq r_i$.

Let $N = (P^0 \cup P_A \cup P_R, T, F)$ be a PLN with $t \in T$. $t^{(p)}$ and $t^{(r)}$ denote the sets of input and output operation places of $t$ respectively, and $t^i$ and $t^r$ denote the sets of input and output resources of $t$ respectively. Hence $t = t^{(p)} \cup t^{(r)}$ and $t^i = t^{(p)} \cup t^{(r)}$.

As shown in Figure 1, $|H(p_{14})| = 2$, $|H(p_{15})| = 2$, $|H(p_{16})| = 2$, $|H(p_{17})| = 2$ and $|H(p_{18})| = 2$. Also we have $|H(p_{14}) \cap P_A| = 1$, $|H(p_{14}) \cap P_A| = 1$, $|H(p_{14}) \cap P_A| = 1$, $|H(p_{15}) \cap P_A| = 0$, $|H(p_{15}) \cap P_A| = 0$, $|H(p_{16}) \cap P_A| = 1$, $|H(p_{16}) \cap P_A| = 1$, $|H(p_{17}) \cap P_A| = 1$, $|H(p_{17}) \cap P_A| = 1$, $|H(p_{18}) \cap P_A| = 1$ and $|H(p_{18}) \cap P_A| = 1$. Moreover, in the two subnets $N_1$ and $N_2$, 

(3) $M(q) \geq 1, \forall r \in P_R$. 

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A Sufficient and Necessary Condition for Reachability in a PLN

Definition 8. Let $N = (P^0 \cup P_A \cup P_R, T, F)$ be a PLN with an admissible initial marking $M_0$. The maximum number of tokens in place $p$ is called the bound of place $p$, denoted by $b_p$. That is to say, $b_p = \max \{ |M(p)| | M \in R(N, M_0) \}$.

Lemma 1 [28]. Let $N = (P^0 \cup P_A \cup P_R, T, F)$ be an LS³PR. The set of minimal P-semiflows of $N$ is $I = I_R \cup I_{SM}$, where $I_R = \bigcup_{r \in P_R} H(r) \cup \{r\}$ and $I_{SM} = \bigcup_{i \in \{1, \ldots, k\}} P_A \cup \{p_0\}$.

Lemma 2. Let $N = (P^0 \cup P_A \cup P_R, T, F)$ be a PLN with an admissible initial marking $M_0$. Then $\forall p^0 \in P^0$, $b_{p^0} = M_0(p^0)$; $\forall r \in P_R$, $b_r = M_0(r)$; and $\forall p \in P_A$, $b_p = M_0(r_p)$.

Proof. From Lemma 1, the set of minimal P-semiflows of a PLN consists of two subsets. The first corresponds to the token conservation law associated with resources. $\forall r \in P_R$, the P-semiflow $I_R = H(r) \cup \{r\}$, states that for each reachable marking $M$, the token conservation law $\sum_{p \in H(r)} M(p) + M(r) = M_0(r)$ is true. The second subset is associated with the token conservation law for each state machine (in the sense of processes). $\forall i \in \{1, \ldots, k\}$, $p_i^0 \in P^0$, the P-semiflow $I_{SM} = P_A \cup \{p_i^0\}$, establishes the invariant relation $\sum_{p \in P_i} M(p) + M(p_i^0) = M_0(p_i^0)$ for each reachable marking $M$. Taking into account of $M \geq 0$, it is easy to see that $\forall p^0 \in P^0$, $b_{p^0} = M_0(p^0)$; $\forall r \in P_R$, $b_r = M_0(r)$; and $\forall p \in P_A$, $b_p = M_0(r_p)$.

Definition 9. Let $N = (P^0 \cup P_A \cup P_R, T, F)$ be a PLN. A circuit $C$ that contains resources and transitions only is called a resource-transition circuit if $C_T = C_T(r) = C_R$, where $C_R$ and $C_T$ denote the sets of all resources and transitions of $C$ respectively.

Definition 10. Let $C(r_1, t_1, r_2, t_2, \ldots, r_m, t_m)$ be a resource-transition circuit in a PLN, where
1) $m \geq 2$; 2) $\forall i \in \{1, 2, \ldots, m\}$, $r_i \in T^*$; 3) $\forall i \in \{2, \ldots, m\}$, $r_{i-1} \in T^*$; and 4) $r_i \in T^*$. $r_i$ is called a connected resource if there exists $r_j = r_j$ in $C(r_1, t_1, r_2, t_2, \ldots, r_m, t_m)$, where $i, j \in \{1, 2, \ldots, m\}$ and $i \neq j$.

As shown in Figure 1, there are three resource-transition circuits in the net: $C_1 = C(p_{14}, t_8, p_{15}, t_2)$, $C_2 = C(p_{15}, t_9, p_{16}, t_3)$ and $C_3 = C(p_{14}, t_8, p_{15}, t_9, p_{16}, t_3, p_{15}, t_2)$ with $(r)C_{i_r} = C_{i_r(r)} = \{p_{14}, p_{15}\} = C_{i_r}$; $(r)C_{r} = C_{r(r)} = \{p_{15}, p_{16}\} = C_{r_2}$ and $(r)C_{r} = C_{r(r)} = \{p_{14}, p_{15}, p_{16}\} = C_{r_3}$. $p_{15}$ appears twice in $C_3$. Hence, $p_{15}$ is a connected resource.

Theorem 1 [11]. Let $N = (P^0 \cup P_A \cup P_R, T, F)$ be a PLN with an admissible initial marking $M_0$. $S = S_A \cup S_R$ is a strict minimal siphon of $N$ if
1) $S_A \neq \phi$, $S_R \neq \phi$;
2) $S_A = C_R$, where $C_R$ is the set of all resources of a resource-transition circuit $C$ in $N$; and
3) $S_A = \{ p | p \in \bigcup_{r \in C_R} H(r) \wedge (p^* \cap (P_A \cup P^0)) \bigcup_{r \in C_R} H(r) \}$.
Corollary 1. Let \( N = (P^0 \cup P_a \cup P_r, T, F) \) be a PLN and \( S = S_d \cup S_r \) be a strict minimal siphon of \( N \). Then \( |S_d| = 2 \).

Proof. From Theorem 1, we have \( S_r = C_r \) and \( S_d = \{ p \mid p \in \bigcup_{r \in C_x} H(r) \land (p^* \cap (P_a \cup P^0) \cup \bigcup_{r \in C_x} H(r)) \} \). We accordingly have the following two cases:

1. \( |S_r| = 2 \). By the definition of resource-transition circuits, there necessarily exist two transitions in \( C_r \) that is the set of all the transitions of resource-transition circuit \( C \) associated with \( S \). By the definition of a PLN, the two transitions in \( C_r \) necessarily belong to two subnets that are mutually reversed. Since \( S_d = \{ p \mid p \in \bigcup_{r \in C_x} H(r) \land (p^* \cap (P_a \cup P^0) \cup \bigcup_{r \in C_x} H(r)) \} \), \( \forall p \in S_d \), \( p \in C_r \) holds. Therefore, \( |S_d| = 2 \) is true.

2. \( |S_r| > 2 \). By the definition of resource-transition circuits, there necessarily exist more than two transitions in \( C_r \). By the definition of a PLN, the transitions in \( C_r \) necessarily belong to two subnets that are mutually reversed. Since \( S_d = \{ p \mid p \in \bigcup_{r \in C_x} H(r) \land (p^* \cap (P_a \cup P^0) \cup \bigcup_{r \in C_x} H(r)) \} \), \( \forall p \in S_d \), \( p \in C_r \) and \( p \notin H(r) \) hold, where \( r \) is a connected resource in \( S_r \). Therefore, \( |S_d| = 2 \) is true.

Definition 11. Let \( N = (P^0 \cup P_a \cup P_r, T, F) \) be a PLN. An initial marking \( M_0 \) is called an appropriate initial marking of \( N \) if

1. \( M_0(p^0) \geq 1, \forall p^0 \in P^0 \);
2. \( M_0(p) = 0, \forall p \in P_a \); and
3. \( \forall r \in P_r; \) if \( r \) is a connected resource, \( M_0(r) = 1 \), otherwise \( M_0(r) \geq 1 \).

Definition 12. The markings in the set of invariant markings \( I_X(N, M_0) \) that are not in the reachability set \( R(N, M_0) \) are called spurious markings.

A backward firing in \( N \) is equivalent to a forward firing in the reverse net \( N' \)[30]. This implies that the directed path in the reachability graph of \( N' \) from \( M' \) to \( M \) is just the reverse path in the reachability graph of \( N \) from \( M \) to \( M' \). Similarly, a spurious marking in \( N \) does not have directed paths from reachable markings and the corresponding marking in \( N' \) does not have directed paths to reachable markings.

Theorem 2. Let \( N = (P^0 \cup P_a \cup P_r, T, F) \) be a PLN with an appropriate initial marking \( M_0 \), \( \Pi \) be the set of strict minimal siphons, and \( S \) be a strict minimal siphon in \( N \). A marking \( M \) is spurious in the set of invariant markings of \( N \) if \( \forall S \subseteq \Pi \), \( M(S_d) = \sum_{p \in S_d} b_p \), where \( S_d \) is the set of operation places of \( S \) and \( b_p \) is the bound of place \( p \).

Proof. 1. We first prove the sufficiency. By Definition 8, \( \forall p \in S_d \), \( b_p = \max \{ M(p) \mid M \in R(N, M_0) \} \) holds. \( \forall S \subseteq \Pi \), \( M(S_d) = \sum_{p \in S_d} b_p \) means that the number of tokens in each operation place of any strict minimal siphon \( S \) reaches its bound at marking \( M \). We have to prove that \( M \) is a spurious marking. That is to say, we need to show that there exist no directed paths from initial marking \( M \) to \( M \) in \( N \). By Corollary 1, \( |S_d| = 2 \). Without loss of
generality, let \( S_d = \{ p_1, p_2 \} \). From Theorem 1, \(|S_R| \geq 2\) holds. We accordingly have the following two cases.

(1) \(|S_R| = 2\). From Theorem 1, \( S_R = \{ r_{p_1}, r_{p_2} \} \) holds. From the proof of Corollary 1, there necessarily exist two transitions in \( C_r \) that is the set of all transitions of resource-transition circuit \( C \) associated with \( S \), and \( \forall p \in S_d, p \in C_r \) holds. Without loss of generality, let \( C_r = \{ t_1, t_2 \}, t_1 \in^* p_1 \), and \( t_2 \in^* p_2 \).

By contradiction, suppose that \( M \) is reachable from \( M_0 \) in \( N \) with \( M(p_1) = M_0(r_{p_1}) \) and \( M(p_2) = M_0(r_{p_2}) \). By Definition 11, \( M_0(p_1) = 0 \) and \( M_0(p_2) = 0 \) hold. According to the token conservation law, \( M(r_{p_1}) = 0 \) and \( M(r_{p_2}) = 0 \) hold. Since \( M \) is reachable from \( M_0 \) in \( N \), there necessarily exists a reachable marking \( M' \) in the reachability graph of \( N \) such that \( M'\{ t_1, t_2 \} M \) or \( M\{ t_2 \} M \) holds. This implies that \( t_2 \) is enabled at \( M \) in the reverse net \( N' \) such that \( M\{ t_2 \} M' \) or \( M\{ t_2 \} M' \). By the definition of a PLN and Definition 10, \( r_{p_1} \in t_1^* \) and \( r_{p_2} \in t_1^* \) hold in \( N \). This implies that \( r_{p_1} \in^* t_2 \) and \( r_{p_2} \in^* t_1 \) hold in the reverse net \( N' \). Therefore, both \( t_1 \) and \( t_2 \) are disabled at \( M \) in the reverse net \( N' \), which contradicts that \( t_1 \) or \( t_2 \) is enabled at \( M \) in the reverse net \( N' \). Thus, \( M \) is a spurious marking in \( N \).

(2) \(|S_R| > 2\). From Theorem 1, there necessarily exist connected resources in \( S_R \). We denote \( R_C \) as the set of connected resources in \( S_R \). From the proof of Corollary 1, there necessarily exist more than two transitions in \( C_r \), and \( \forall p \in S_d, p \in C_r \) and \( p \not\in H(r_i) \) hold, where \( r_i \in R_C \). Let \( t_1 \in^* p_1 \) and \( t_2 \in^* p_2 \). From Theorem 1, \( r_{p_1} \not\in R_C \) and \( r_{p_2} \not\in R_C \) hold. Since \( M_0 \) is an appropriate initial marking, \( \forall r_c \in R_C, M_0(r_c) = 1 \) holds.

By contradiction, suppose that \( M \) is reachable from \( M_0 \) in \( N \) with \( M(p_1) = M_0(r_{p_1}) \) and \( M(p_2) = M_0(r_{p_2}) \). From the proof of case (1), since there exist connected resources in \( S_R \), \( \forall r_c \in R_C, M(r_c) = 1 \) may hold according to the token conservation law. Therefore, \( t_1 \) or \( t_2 \) may be enabled at \( M \) in the reverse net \( N' \). Similarly, there exist a sequence of transitions \( \sigma \) in \( R_C^* \), which may be enabled from \( M \) in \( N' \) such that \( M(\sigma) M^* \) with \( M^*(r_{p_2}) = 0 \). Therefore, the transition \( t \in r_{p_2}^* \) must be disabled at \( M^* \). That is to say, \( M \) does not have directed path to \( M_0 \) in \( N' \). This implies \( M_0 \) does not have directed path to \( M \) in \( N \), which contradicts that \( M \) is reachable in \( N \). Therefore, we can conclude that \( M \) is a spurious marking.

2. We prove the necessity. By contradiction, suppose that \( \forall S \subseteq \Pi, M(S_d) \not\in \sum_{p \in S_d} b_p \). Since \( M \) is a marking in the set of invariant markings of \( N \), the token conservation law \( \sum_{p \in H(r)} M(p) + M(r) = M_0(r) \) is true. From Lemma 2, \( \forall p \in S_d, b_p = M_0(r_p) \) holds. Note that \( M \geq 0 \). We can conclude that \( \forall p \in S_d, M(p) \leq b_p \). Therefore, \( M(S_d) < \sum_{p \in S_d} b_p \) holds.

\( M(S_d) < \sum_{p \in S_d} b_p \) means that at marking \( M \) the number of tokens in each operation place of \( S \) does not reach its bound at the same time. From the proof of the sufficiency, we can similarly prove that \( M \) is reachable from \( M_0 \), which contradicts that \( M \) is a spurious marking. Therefore, if \( M \) is a spurious marking in the set of invariant markings of \( N \), \( \forall S \subseteq \Pi, M(S_d) = \sum_{p \in S_d} b_p \) holds.
An Algorithm Computing the Set of Reachable Markings

From Theorem 2, given a PLN with an appropriate initial marking \( M_0 \), all the spurious markings can be identified from the set of invariant markings. Then the set of reachable markings can be calculated by removing all the spurious markings from the set of invariant markings. An algorithm to compute the set of markings reachable from \( M_0 \) is presented as follows:

**Algorithm 1.** Computation of reachable markings

Input: a PLN model \((N,M_0)\).
Output: The reachability set \( R(N,M_0) \).
1) Check if \( N \) is a PLN and \( M_0 \) is an appropriate initial marking. If not, exit.
2) Compute the set of minimal \( P \)-semiflows by Lemma 1.
3) Compute the bounds of all the places by Lemma 2. According to the token conservation law, enumerate the set of invariant markings \( I_x(N,M_0) = \{ M \in \mathrm{IN}^P | X^TM = X^TM_0 \} \), where \( X \) is a matrix whose column each is a \( P \)-semiflow of the net \((N,M_0)\), and \( \mathrm{IN}^P \) is a set of non-negative vectors with a length of \(|P|\).
4) Compute the set \( C \) of resource-transition circuits by Definition 9.
5) Compute the set of strict minimal siphons \( \Pi \) due to Theorem 1.
6) If \( \{ \Pi = \emptyset \} \) then \( R(N,M_0) = I_x(N,M_0) \).
   else \( R(N,M_0) = I_x(N,M_0) \setminus \{ M | \forall S \subseteq \Pi, M(S_x) = \sum_{p \in S} b_p \} \).
7) Output \( R(N,M_0) \).
8) End.

**AN EXAMPLE**

To practically test the efficiency of the proposed method, a C program has been developed, which implements the algorithm and runs on a Windows XP operating system with Intel CPU Core 2.60 GHz and 3 GB memory.

Take the net in Figure 1 as an example. There are 18 places and 13 transitions. It is a PLN and \( M_0 \) is an appropriate initial marking. By Lemma 1, the net has eight minimal \( P \)-semiflows as follows:

\[
I_{SM_1} = \{p_1, p_2, p_3, p_4, p_5, p_6\} , \quad I_{SM_2} = \{p_7, p_8, p_9, p_{10}\} , \quad I_{SM_3} = \{p_{11}, p_{12}, p_{13}\} , \\
I_h = \{p_2, p_8, p_{14}\} , \quad I_s = \{p_3, p_9, p_{15}\} , \quad I_t = \{p_4, p_{10}, p_{16}\} , \quad I_m = \{p_5, p_{12}, p_{17}\} \quad \text{and} \quad I_r = \{p_6, p_{13}, p_{18}\}.
\]

By Lemma 2, the bounds of all the places are as follows: \( b_{p_1} = 8 \), \( b_{p_2} = 2 \), \( b_{p_3} = 1 \), \( b_{p_4} = 2 \), \( b_{p_5} = 1 \), \( b_{p_6} = 2 \), \( b_{p_7} = 5 \), \( b_{p_8} = 2 \), \( b_{p_9} = 1 \), \( b_{p_{10}} = 2 \), \( b_{p_{11}} = 3 \), \( b_{p_{12}} = 1 \), \( b_{p_{13}} = 2 \), \( b_{p_{14}} = 2 \), \( b_{p_{15}} = 1 \), \( b_{p_{16}} = 2 \), \( b_{p_{17}} = 1 \) and \( b_{p_{18}} = 2 \). By the definition of the set invariant markings \( I_x(N,M_0) \), we can obtain \( I_x(N,M_0) \) that has 1944 markings.

By Definition 8, the net has three resource-transition circuits: \( C_1 = C(p_{14}, p_8, p_{15}, t_2) \), \( C_2 = C(p_{15}, t_9, p_{16}, t_3) \) and \( C_3 = C(p_{14}, t_8, p_{15}, t_9, p_{16}, t_3, p_{15}, t_2) \). By Theorem 1 we can find that the net correspondingly has three strict minimal siphons: \( S_1 = \{p_3, p_8, p_{14}, p_{15}\} , \quad S_2 = \{p_4, p_9, p_{15}, p_{16}\} \) and \( S_3 = \{p_4, p_9, p_{14}, p_{15}, p_{16}\} \).

By Theorem 2, a marking \( M \) with \( M(p_1) = 2 \) and \( M(p_8) = 1 \) is spurious in \( I_x(N,M_0) \), a marking \( M' \) with \( M'(p_1) = 2 \) and \( M'(p_8) = 1 \) is spurious, and a marking \( M'' \) with \( M''(p_4) = 2 \) and \( M''(p_8) = 2 \) is also spurious. We impose the constraints on \( I_x(N,M_0) \) as follows:
\( M(p_{x}) + M(p_{y}) < 3 \), \( M(p_{x}) + M(p_{y}) < 3 \) and \( M(p_{x}) + M(p_{y}) < 4 \), where \( M \in I_{\lambda}(N, M_{0}) \). Then the reachability set \( R(N, M_{0}) \) that has 1710 markings is generated by removing 234 spurious ones from \( I_{\lambda}(N, M_{0}) \).

The software package INA2003 [30] can also compute the reachability set. For comparison, reachability analysis of the Petri net in Figure 1 is conducted by the use of INA. The tool generates the reachability set consisting of 1710 markings, which are in agreement with the markings in the reachability set \( R(N, M_{0}) \) generated by the proposed method, validating the correctness of the proposed algorithm.

TINA [31] is a toolbox for editing and analyzing Petri nets, which can also compute the reachability set. For comparison, reachability analysis of the Petri net in Figure 1 is conducted by the use of TINA. The toolbox generates a reachability set consisting of 1710 markings, which are also in agreement with the markings in the reachability set \( R(N, M_{0}) \) generated by the proposed method.

**EXPERIMENTAL RESULTS**

The net structure in Figure 1 is selected for experimental studies. We vary the initial markings of resource places \( p_{14} \), \( p_{15} \), \( p_{16} \), \( p_{17} \) and \( p_{18} \), and idle places \( p_{1} \), \( p_{7} \) and \( p_{12} \). Table 1 shows various parameters in the net, where the first column represents the initial tokens in places \( p_{1} \), \( p_{7} \), \( p_{12} \), \( p_{14} \), \( p_{15} \), \( p_{16} \), \( p_{17} \) and \( p_{18} \). \( N_{I} \), \( N_{S} \) and \( N_{R} \) indicate the numbers of invariant markings, spurious markings and reachable markings respectively. The fifth column shows the total CPU time for computing \( R(N, M_{0}) \) by using the proposed method. The sixth and the last columns show the total CPU time for computing \( R(N, M_{0}) \) by using INA and the total CPU time for computing \( R(N, M_{0}) \) by using TINA for comparison purpose respectively.

<table>
<thead>
<tr>
<th>( p_{1} ), ( p_{7} ), ( p_{12} ), ( p_{14} ), ( p_{15} ), ( p_{16} ), ( p_{17} ), ( p_{18} )</th>
<th>( N_{I} )</th>
<th>( N_{S} )</th>
<th>( N_{R} )</th>
<th>CPU time (s)</th>
<th>INA time (s)</th>
<th>TINA time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8, 5, 3, 2, 1, 2, 1, 2</td>
<td>1,944</td>
<td>234</td>
<td>1,710</td>
<td>&lt;1</td>
<td>&lt;1</td>
<td>&lt;1</td>
</tr>
<tr>
<td>15, 9, 6, 4, 1, 4, 2, 4</td>
<td>60,750</td>
<td>2,790</td>
<td>57,960</td>
<td>&lt;1</td>
<td>216</td>
<td>3</td>
</tr>
<tr>
<td>29, 17, 12, 8, 1, 8, 4, 8</td>
<td>4,100,625</td>
<td>61,425</td>
<td>4,039,200</td>
<td>22</td>
<td>&gt;7200</td>
<td>—</td>
</tr>
<tr>
<td>36, 21, 15, 10, 1, 10, 5, 10</td>
<td>18,112,248</td>
<td>184,338</td>
<td>17,927,910</td>
<td>398</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>43, 25, 18, 12, 1, 12, 6, 12</td>
<td>63,299,964</td>
<td>466,284</td>
<td>62,833,680</td>
<td>1432</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>50, 29, 21, 14, 1, 14, 7, 14</td>
<td>186,624,000</td>
<td>1,041,120</td>
<td>185,582,880</td>
<td>5365</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

As shown in Table 1, we can see that the proposed method becomes more efficient with the increase of the initial markings. Note that “—” in Table 1 means that the computation cannot be finished with a reasonable time or memory is overflowed.
CONCLUSIONS

The set of reachable markings play an important role in the deadlock control in Petri nets. This paper presents a novel approach in computing the set of reachable markings using P-invariants and strict minimal siphons without the construction of reachability graph that often makes the analysis intractable. The method is applied to a small class of Petri nets called PLNs that are a subclass of LS³PR. Experimental results show its efficiency via studying a number of examples. Future work should extend the method in this paper to more general classes of Petri nets.

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