Communication

On the Diophantine equation $2^x + 11^y = z^2$

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Abstract: In this paper it is shown that $(3,0,3)$ is the only non-negative integer solution of the Diophantine equation $2^x + 11^y = z^2$.

Keywords: exponential Diophantine equation, Catalan’s conjecture

INTRODUCTION

Solving Diophantine equations of the form $2^x + p^y = z^2$, where $p$ is prime, has been widely studied by many mathematicians. For example, Acu [1] proved in 2007 that $(x, y, z) = (3,0,3), (2,1,3)$ are the only two non-negative solutions for the case $p = 5$. On the other hand, it was shown in 2010 by Suvarnamani et al. [2] that there are no non-negative solution $(x, y, z)$, with $x$ even, for the case $p = 7, 11$. The complete sets of non-negative solutions to some Diophantine equations of similar forms were also studied by Sándor in 2002 [3].

Later in 2011, Suvarnamani [4] published a paper on finding non-negative solutions to the Diophantine equations of the form $2^x + p^y = z^2$ for every prime $p$. Having this result, it might seem at first glance that the problem of solving the Diophantine equations of such form did come to an end. Nevertheless, for the case where $x$ is even (i.e. $x = 2m$ for some integer $m$), these equations can be rewritten as $4^m + p^y = z^2$, whose complete set of non-negative solutions for each prime $p$ was readily found by the author [5]. In addition, for the case where $x$ is odd, Suvarnamani’s proof [4] unfortunately contains a misleading argument which significantly affects its correctness: it was stated (p.1417, line 14-15) that

$$(z - 2^{k+\frac{1}{2}})(z + 2^{k+\frac{1}{2}}) = p^y,$$

for some integer $k, p, y, z$, and thus,

$$z - 2^{k+\frac{1}{2}} = p^y \text{ and } z + 2^{k+\frac{1}{2}} = p^{y-u},$$

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for some integer \( u \geq 0 \). This is clearly absurd since \( z + 2^{k + \frac{1}{2}} \) is irrational while \( p^u, p^{y-u} \) are integers. Since no extra information can be obtained from Suvarnamani’s proof, solving the Diophantine equation of the form \( 2^x + p^y = z^2 \), where \( p \) is prime, now remains an open problem.

Inspired by all the aforementioned results, this paper therefore aims to study the Diophantine equation \( 2^x + p^y = z^2 \), particularly where \( p = 11 \), in detail. To be precise, the objective is to show that \( (x,y,z) = (3,0,3) \), which clearly satisfies such equation, is its only non-negative solution.

**MAIN RESULTS**

In this study, Catalan’s conjecture [6], which states that the only solution in integers \( a > 1, b > 1, x > 1, y > 1 \) of the equation \( a^x + b^y = 1 \) is \( (a,b,x,y) = (3,2,2,3) \), is used.

In the main theorem, the Diophantine equation \( 2^x + 11^y = z^2 \) is considered.

**Theorem.** The Diophantine equation \( 2^x + 11^y = z^2 \) has only one solution in non-negative integer, namely \( (x,y,z) = (3,0,3) \).

**Proof.** Consider the following cases:

Case 1: \( x = 0 \). It can be easily checked that if the Diophantine equation \( 1 + 11^y = z^2 \) has a solution, then \( y \geq 2 \) and \( z \) is an even integer greater than 3. Thus,

\[
11^y = z^2 - 1 = (z + 1)(z - 1).
\]

Then there are non-negative integers \( \alpha, \beta \) such that \( 11^\alpha = z + 1, 11^\beta = z - 1 \), \( \alpha > \beta \) and \( \alpha + \beta = y \). Therefore,

\[
11^\beta (11^{\alpha - \beta} - 1) = 11^\alpha - 11^\beta = (z + 1) - (z - 1) = 2,
\]

which implies that \( \beta = 0 \) and \( 11^\alpha - 1 = 2 \). This contradicts the fact that \( \alpha \) is a non-negative integer. Therefore, the Diophantine equation \( 1 + 11^y = z^2 \) has no solution.

Case 2: \( x = 1 \). If the Diophantine equation \( 2 + 11^y = z^2 \) has a solution, then this implies that \( z^2 \equiv 2 \) (mod 11) has a solution. However, it is easy to check that \( z^2 \equiv 2 \) (mod 11) does not have a solution, a contradiction. Thus, the Diophantine equation \( 2 + 11^y = z^2 \) also has no solution.

Case 3: \( x \geq 2 \). Therefore \( 2^x \equiv 0 \) (mod 4) and \( z^2 \equiv 0, 1 \) (mod 4), which implies that \( 11^y \equiv 0, 1 \) (mod 4). It can be observed that if \( y \) is an odd non-negative integer, then \( 11^y \equiv 3 \) (mod 4). Therefore, \( y \) is an even non-negative integer. Let \( y = 2k \) for some non-negative integer \( k \). Thus,

\[
2^x = z^2 - 11^{2k} = (z + 11^k)(z - 11^k).
\]

Then there are non-negative integers \( \alpha, \beta \) such that \( 2^\alpha = z + 11^k, 2^\beta = z - 11^k \) with \( \alpha > \beta \) and \( \alpha + \beta = x \). Therefore,

\[
2^\beta (2^{\alpha - \beta} - 1) = 2^\alpha - 2^\beta = (z + 11^k) - (z - 11^k) = 2(11^k).
\]

This implies that \( \beta = 1 \) and

\[
2^{\alpha - 1} - 1 = 11^k. \tag{*}
\]

So \( z = 11^k + 2 \) and \( \alpha > \beta = 1 \). By Catalan’s conjecture, \( 2^{\alpha - 1} - 1 = 11^k \) has no solution only when \( \alpha - 1 > 1 \) and \( k > 1 \). Thus, it suffices to consider only the case when \( \alpha - 1 \leq 1 \) or \( k \leq 1 \), i.e. \( \alpha = 2 \) or \( 0 \leq k \leq 1 \). From (*), it is easy to see that \( \alpha = 2 \) if and only if \( k = 0 \). This implies that
(x, y, z) = (3, 0, 3). Finally, one can easily check that $2^{x-1} - 1 = 11^k$ does not have a solution when $k = 1$.

It is easy to check that $(x, y, z) = (3, 0, 3)$ is the only non-negative integer solution of $2^x + 11^y = z^2$. This finishes the proof. □

OPEN PROBLEM

It is to be noted that all finding of the solutions of Diophantine equation in the case of $2^x + p^y = z^2$ where $p$ is prime in general is still an open problem. For example, it is not known how to find all non-negative integer solutions of $2^x + p^y = z^2$ where $p = 7, 13, 29, 37$ or 257.

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REFERENCES


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