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Communication

On the Diophantine equation $2^x + 11^y = z^2$

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Abstract: In this paper it is shown that (3,0,3) is the only non-negative integer solution of the Diophantine equation $2^x + 11^y = z^2$.

Keywords: exponential Diophantine equation, Catalan's conjecture

INTRODUCTION

Solving Diophantine equations of the form $2^x + p^y = z^2$, where p is prime, has been widely studied by many mathematicians. For example, Acu [1] proved in 2007 that $(x, y, z) = (3,0,3)$, $(2,1,3)$ are the only two non-negative solutions for the case $p = 5$. On the other hand, it was shown in 2010 by Suvarnamani et al. [2] that there are no non-negative solution (x, y, z), with x even, for the case $p = 7, 11$. The complete sets of non-negative solutions to some Diophantine equations of similar forms were also studied by Sándor in 2002 [3].

Later in 2011, Suvarnamani [4] published a paper on finding non-negative solutions to the Diophantine equations of the form $2^x + p^y = z^2$ for every prime p. Having this result, it might seem at first glance that the problem of solving the Diophantine equations of such form did come to an end. Nevertheless, for the case where x is even (i.e. $x = 2m$ for some integer m), these equations can be rewritten as $4^m + p^y = z^2$, whose complete set of non-negative solutions for each prime p was readily found by the author [5]. In addition, for the case where x is odd, Suvarnamani's proof [4] unfortunately contains a misleading argument which significantly affects its correctness: it was stated (p.1417, line 14-15) that

$$(z - 2^{k+\frac{1}{2}})(z + 2^{k+\frac{1}{2}}) = p^y,$$

for some integer k, p, y, z, and thus,

$$z - 2^{k+\frac{1}{2}} = p^u \text{ and } z + 2^{k+\frac{1}{2}} = p^{y-u},$$

Maejo Int. J. Sci. Technol. **2013**, *7*(02), 291-293

for some integer $u \geq 0$. This is clearly absurd since $z \pm 2^{k+\frac{1}{2}}$ is irrational while p^u, p^{y-u} are integers. Since no extra information can be obtained from Suvarnamani's proof, solving the Diophantine equation of the form $2^x + p^y = z^2$, where *p* is prime, now remains an open problem.

Inspired by all the aforementioned results, this paper therefore aims to study the Diophantine equation $2^x + p^y = z^2$, particularly where $p = 11$, in detail. To be precise, the objective is to show that $(x, y, z) = (3,0,3)$, which clearly satisfies such equation, is its only non-negative solution.

MAIN RESULTS

In this study, Catalan's conjecture [6], which states that the only solution in integers $a > 1$, $b > 1, x > 1, y > 1$ of the equation $a^x + b^y = 1$ is $(a,b,x,y) = (3,2,2,3)$, is used.

In the main theorem, the Diophantine equation $2^x + 11^y = z^2$ is considered.

Theorem. *The Diophantine equation* $2^x + 11^y = z^2$ *has only one solution in non-negative integer, namely* $(x, y, z) = (3,0,3)$.

Proof. Consider the following cases:

Case 1: $x = 0$. It can be easily checked that if the Diophantine equation $1 + 11^y = z^2$ has a solution, then $y \geq 2$ and z is an even integer greater than 3. Thus,

$$11^y = z^2 - 1 = (z+1)(z-1).$$

Then there are non-negative integers α, β such that $11^\alpha = z+1$, $11^\beta = z-1$, $\alpha > \beta$ and $\alpha + \beta = y$. Therefore,

$$11^\beta (11^{\alpha-\beta} - 1) = 11^\alpha - 11^\beta = (z+1) - (z-1) = 2,$$

which implies that $\beta = 0$ and $11^\alpha - 1 = 2$. This contradicts the fact that α is a non-negative integer. Therefore, the Diophantine equation $1 + 11^y = z^2$ has no solution.

Case 2: $x = 1$. If the Diophantine equation $2 + 11^y = z^2$ has a solution, then this implies that $z^2 \equiv 2 \pmod{11}$ has a solution. However, it is easy to check that $z^2 \equiv 2 \pmod{11}$ does not have a solution, a contradiction. Thus, the Diophantine equation $2 + 11^y = z^2$ also has no solution.

Case 3: $x \geq 2$. Therefore $2^x \equiv 0 \pmod 4$ and $z^2 \equiv 0, 1 \pmod 4$, which implies that $11^y \equiv 0, 1 \pmod 4$. It can be observed that if y is an odd non-negative integer, then $11^y \equiv 3 \pmod 4$. Therefore, *y* is an even non-negative integer. Let $y = 2k$ for some non-negative integer *k*. Thus,

$$2^x = z^2 - 11^{2k} = (z + 11^k)(z - 11^k).$$

Then there are non-negative integers α, β such that $2^\alpha = z + 11^k$, $2^\beta = z - 11^k$ with $\alpha > \beta$ and $\alpha + \beta = x$. Therefore,

$$2^\beta (2^{\alpha-\beta} - 1) = 2^\alpha - 2^\beta = (z + 11^k) - (z - 11^k) = 2(11^k).$$

This implies that $\beta = 1$ and

$$2^{\alpha-1} - 1 = 11^k. \qquad (*)$$

So $z = 11^k + 2$ and $\alpha > \beta = 1$. By Catalan's conjecture, $2^{\alpha-1} - 1 = 11^k$ has no solution only when $\alpha - 1 > 1$ and $k > 1$. Thus, it suffices to consider only the case when $\alpha - 1 \leq 1$ or $k \leq 1$, i.e. $\alpha = 2$ or $0 \leq k \leq 1$. From (*), it is easy to see that $\alpha = 2$ if and only if $k = 0$. This implies that

Maejo Int. J. Sci. Technol. **2013**, *7*(02), 291-293

$(x, y, z) = (3,0,3)$. Finally, one can easily check that $2^{\alpha-1} - 1 = 11^k$ dose not have a solution when $k = 1$.

It is easy to check that $(x, y, z) = (3,0,3)$ is the only non-negative integer solution of $2^x + 11^y = z^2$. This finishes the proof. □

OPEN PROBLEM

It is to be noted that all finding of the solutions of Diophantine equation in the case of $2^x + p^y = z^2$ where p is prime in general is still an open problem. For example, it is not known how to find all non-negative integer solutions of $2^x + p^y = z^2$ where p = 7, 13, 29, 37 or 257.

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