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On certain Banach spaces of difference sequences of fuzzy real numbers

Hemen Dutta

Department of Mathematics, Gauhati University, Guwahati-781014, India

E-mail: hemen_dutta08@rediffmail.com

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Abstract: Some sets of difference sequences of fuzzy real numbers which have Banach subspaces are introduced and investigated. Further, it is shown that these subspaces are isometric with some subsets of the sets of null, convergent and bounded sequences of fuzzy real numbers.

Keywords: fuzzy real numbers, sequence of fuzzy real numbers, difference sequences, isometry

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INTRODUCTION

The concept of fuzzy sets and fuzzy set operations was first introduced by Zadeh [1] and subsequently several authors have studied various aspects of the theory and applications of fuzzy sets. Bounded and convergent sequences of fuzzy numbers were introduced by Matloka [2], who showed that every convergent sequence is bounded. Nanda [3] studied the spaces of bounded and convergent sequence of fuzzy numbers and showed that they are complete metric spaces. Çakan and Şengönül [4] studied the space of convergent and bounded sequence of fuzzy numbers and showed that they are Banach spaces under a suitable norm. Altmok *et al.* [5] introduced the notions of lacunary almost statistical convergence and lacunary strongly almost convergent sequences. In comparison with the classical sequence spaces, several authors have extended such notions to sequences of fuzzy numbers. One may see also Savaş [6] for sequence of fuzzy numbers.

The notion of difference sequence of complex numbers was introduced by Kizmaz [7]. Later on Et and Colak [8], Tripathy and Et [9], Tripathy *et al.* [10], Dutta [11-13], Dutta and Bilgin [14], Başarır *et al*. [15] and some others have defined difference sequences by introducing different difference operators. In this article the author uses some difference operators which generalise all previous such operators and extends the notion of difference sequences to sequences of fuzzy numbers using these most generalised difference operators. First, some well known definitions are given.

Let D denote a set of all closed bounded intervals $A = \left[A_1, A_2\right]$ on real line R. Let A, B be two closed bounded intervals (i.e. $A, B \in D$). Then we define:

$A \leq B$ if and only if $A_1 \leq B_1$ and $A_2 \leq B_2$, and
$d(A, B) = \max(|A_1\text{-}B_1|, |A_2\text{-}B_2|)$.

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Then d is a metric on D. It is well known that (D, d) is a complete metric space and '$\leq$' is a partial order relation in D.

A *fuzzy real number* X is a fuzzy set on R, i.e. a mapping $X : R \to I$ (=[0,1]) associating each real number t with its grade of membership $X(t)$. A fuzzy real number X is called *convex* if $X(t) \geq X(s) \wedge X(r) =$ min ($X(s), X(t)$), where $s < t < r$. If there exists $t_0 \in R$ such that $X(t_0)$ =1, then the fuzzy real number X is called *normal.* A fuzzy real number X is said to be *upper-semi-continuous* if, for each ε>0, X^{-1}([0, $a + \varepsilon$)) for all $a \in I$ is open in the usual topology of R. The set of all upper-semicontinuous, normal, convex fuzzy real numbers is denoted by $R(I)$. Throughout the article, by a fuzzy real number we mean that the number belongs to $R(I)$.

The *α-level* set $[X]^{\alpha}$ of the fuzzy real number X, for $0 < \alpha \leq 1$, is defined as $[X]^{\alpha} = \{t \in R : X(t) \geq \alpha\}$; for $\alpha = 0$, it is the closure of the strong 0-cut. Throughout the article α means $\alpha \in [0, 1]$ unless otherwise stated.

A fuzzy real number X is called *non-negative* if $X(t) = 0$ for all $t < 0$. The set of all non-negative fuzzy real numbers is denoted by $R^*(I)$.

Let $\bar{d} : R(I) \times R(I) \to R$ be defined by:

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d\left([X]^{\alpha}, [Y]^{\alpha}\right).$$

Then $\bar{d}$ defines a metric on $R(I)$. In fact ($R(I)$, $\bar{d}$) is a complete metric space. The additive identity and multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

For any two elements X, Y of $R(I)$, let us define:

$X \leq Y$ if and only if $[X]^{\alpha} \leq [Y]^{\alpha}$ for any $\alpha \in [0, 1]$.

A subset E of $R(I)$ is said to be bounded above if there exists a fuzzy real number M, called an upper bound of E, such that $X \leq M$ for every $X \in E$. M is called the least upper bound or supremum of E if M is an upper bound and the smallest of all upper bounds. A lower bound and the greatest lower bound or infimum are defined similarly. E is said to be bounded if it is bounded both above and below.

If (M, ρ) is a linear metric space, then it is known that $g(x) = \rho(x, \theta)$ is a norm on M, where θ is the zero element in M.

Now let $\bar{\theta}$ be a fuzzy number such that $\bar{\theta}(x) = 0$ for all $x \in R$; then

$$\|X\| = \bar{d}\left(X, \bar{\theta}\right)$$

is a norm on $R(I)$. Also, it is easy to show that $R(I)$ is a complete normed linear space, i.e. the space is a Banach space.

A sequence $X = (X_k)$ of fuzzy real numbers is a function X from the set N of all positive integers into $R(I)$. The fuzzy real number X_k denotes the value of the function at $k \in N$ and is called the k^{th} term or general term of the sequence.

The set of convergent sequences is denoted by c^F. A sequence $X = (X_k)$ of fuzzy numbers is said to be convergent to the fuzzy real number X_0, written as $\lim_k X_k = X_0$, if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\|X_k - X_0\| < \varepsilon, \text{ for } k \geq n_0 .$$

A sequence $X = (X_k)$ of fuzzy real numbers is said to be a Cauchy sequence if, for every $\varepsilon > 0$, there exists $n_0 \in N$ such that

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$$\|X_k - X_l\| < \varepsilon, \text{ for } k, l \geq n_0 .$$

The set of bounded sequences is denoted by ℓ_∞^F. A sequence $X = (X_k)$ of fuzzy real numbers is said to be bounded if the set $\{X_k: k \in N\}$ of fuzzy numbers is bounded; equivalently,

$$\sup_k \|X_k\| < \infty .$$

NEW DEFINITIONS AND RESULTS

Let r and s be two non-negative integers and $v = (v_k)$ be a sequence of non-zero reals. Then we define the following new definitions and spaces.

A sequence $X = (X_k)$ of fuzzy numbers is said to be $\Delta^s_{(v,r)}$-convergent to the fuzzy real number X_0, written as $\lim_k \Delta^s_{(v,r)} X_k = X_0$, if for every $\varepsilon > 0$ there exists $n_0 \in N$ such that

$$\left\|\Delta^s_{(v,r)} X_k - X_0\right\| < \varepsilon \text{ for } k \geq n_0,$$

where $\left(\Delta^s_{(v,r)} X_k\right) = \left(\Delta^{s-1}_{(v,r)} X_k - \Delta^{s-1}_{(v,r)} X_{k-r}\right)$ and $\Delta^0_{(v,r)} X_k = v_k X_k$ for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta^s_{(v,r)} X_k = \sum_{i=0}^{s} (-1)^i \binom{s}{i} v_{k-ri} X_{k-ri} .$$

In this expansion, we take $v_k = 0$ and $X_k = \bar{0}$ for non-positive values of k. (1)

Let $c^F\left(\Delta^s_{(v,r)}\right)$ denote the set of all $\Delta^s_{(v,r)}$-convergent sequences of fuzzy real numbers. In particular if $X_0 = \bar{0}$ in the above definition, we say $X = (X_k)$ to be $\Delta^s_{(v,r)}$-null sequence of fuzzy real numbers and we denote the set of all $\Delta^s_{(v,r)}$-null sequences of fuzzy real numbers by $c_0^F\left(\Delta^s_{(v,r)}\right)$.

A sequence $X = (X_k)$ of fuzzy numbers is said to be $\Delta^s_{(v,r)}$-bounded if the set $\{\Delta^s_{(v,r)} X_k : k \in N\}$ of fuzzy real numbers is bounded.

Let $\ell_\infty^F\left(\Delta^s_{(v,r)}\right)$ denote the set of all $\Delta^s_{(v,r)}$-bounded sequences of fuzzy real numbers.

Similarly we can define the sets $c_0^F\left(\Delta^s_{v,r}\right)$, $c^F\left(\Delta^s_{v,r}\right)$ and $\ell_\infty^F\left(\Delta^s_{v,r}\right)$ of $\Delta^s_{v,r}$-null, $\Delta^s_{v,r}$-convergent and $\Delta^s_{v,r}$-bounded sequences of fuzzy real numbers, where $\left(\Delta^s_{v,r} X_k\right) = \left(\Delta^{s-1}_{v,r} X_k - \Delta^{s-1}_{v,r} X_{k+r}\right)$ and $\Delta^0_{v,r} X_k = v_k X_k$ for all $k \in N$, which is equivalent to the following binomial representation:

$$\Delta^s_{v,r} X_k = \sum_{i=0}^{s} (-1)^i \binom{s}{i} v_{k+ri} X_{k+ri} .$$

Taking $s = 0$ and $v_k = 1$ for all $k \in N$ in the above definitions, we get the spaces c_0^F, c^F and ℓ_∞^F.

In general, the spaces $c_0^F\left(\Delta^s_{v,r}\right)$, $c^F\left(\Delta^s_{v,r}\right)$ and $\ell_\infty^F\left(\Delta^s_{v,r}\right)$ are not Banach spaces. In fact, it is not possible in general to find some fuzzy real number X-Y such that X=Y+(X-Y) (called the Hukuhara difference when it exists.)

Let $Sc_0^F\left(\Delta^s_{v,r}\right)$, $Sc^F\left(\Delta^s_{v,r}\right)$ and $S\ell_\infty^F\left(\Delta^s_{v,r}\right)$ be the subsets of $c_0^F\left(\Delta^s_{v,r}\right)$, $c^F\left(\Delta^s_{v,r}\right)$ and $\ell_\infty^F\left(\Delta^s_{v,r}\right)$ respectively, consisting of sequences of fuzzy real numbers that satisfy the Hukuhara difference. Such subsets exist as every real number is a fuzzy real number.

Theorem 1. *The spaces* $Sc_0^F\left(\Delta_{v,r}^s\right)$, $Sc^F\left(\Delta_{v,r}^s\right)$ *and* $S\ell_\infty^F\left(\Delta_{v,r}^s\right)$ *are Banach spaces under the norm:*

$$\|X\|^{/} = \sup_k \left\|\Delta_{(v,r)}^s X_k\right\| \qquad (2)$$

Proof. We prove the result only for the case $Sc^F\left(\Delta_{v,r}^s\right)$; for the other cases it will follow on applying similar arguments. It is easy to see that $\|.\|^{/}$ is a norm on $Sc^F\left(\Delta_{v,r}^s\right)$. To prove completeness, let (X^i) be a Cauchy sequence in $Sc^F\left(\Delta_{v,r}^s\right)$, where $X^i = \left(X_k^i\right) = \left(X_1^i, X_2^i, \ldots\right)$ for each $i \in N$. Then for a given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\left\|X^i - X^j\right\|^{/} = \sup_k \left\|\Delta_{(v,r)}^s X_k^i - \Delta_{(v,r)}^s X_k^j\right\| < \varepsilon \text{ for all } i, j \geq n_0 .$$

It follows that

$$\left\|\Delta_{(v,r)}^s X_k^i - \Delta_{(v,r)}^s X_k^j\right\| < \varepsilon \text{ for all } i, j \geq n_0 \text{ and } k \in N .$$

This implies that $\left(\Delta_{(v,r)}^s X_k^i\right)$ is a Cauchy sequence in $L(R)$ for all $k \geq 1$. But $L(R)$ is complete, so $\left(\Delta_{(v,r)}^s X_k^i\right)$ is convergent in $L(R)$ for all $k \geq 1$.

For simplicity, let $\lim_{i\to\infty} \Delta_{(v,r)}^s X_k^i = \sum_{u=0}^{s} (-1)^u \binom{s}{u} v_{k-ru} X_{k-ru}^i = N_k$, for example for each $k \geq 1$. Considering $k = 1, 2, \ldots rs, \ldots$, we can easily conclude that $\lim_{i\to\infty} X_k^i = X_k$ exists for each $k \geq 1$ (see (1)).

It remains to show that $X = (X_k) \in Sc^F\left(\Delta_{v,r}^s\right)$. Now we can find that

$$\lim_{j\to\infty} \left\|\Delta_{(v,r)}^s X_k^i - \Delta_{(v,r)}^s X_k^j\right\| < \varepsilon \text{ for all } i \geq n_0 \text{ and } k \in N .$$

Hence,

$$\left\|\Delta_{(v,r)}^s X_k^i - \Delta_{(v,r)}^s X_k\right\| < \varepsilon \text{ for all } i \geq n_0 \text{ and } k \in N .$$

This implies that

$$\left\|X^i - X\right\|^{/} < \varepsilon \text{ for all } i \geq n_0$$

Since $Sc^F\left(\Delta_{v,r}^s\right)$ is a linear space, it follows that $X = (X_k) \in Sc^F\left(\Delta_{v,r}^s\right)$. This completes the proof.

Theorem 2. *The spaces* $Sc_0^F\left(\Delta_{v,r}^s\right)$, $Sc^F\left(\Delta_{v,r}^s\right)$ and $S\ell_\infty^F\left(\Delta_{v,r}^s\right)$ *are Banach spaces under the norm:*

$$\|X\|^{//} = \sum_{k=1}^{rs} \|X_k\| + \sup_k \left\|\Delta_{v,r}^s X_k\right\|$$

Proof. The proof follows by similar arguments as applied to proving Theorem 1.

Remark 1. For any sequence $X = (X_k)$, $X \in Z\left(\Delta_{v,r}^s\right)$ if and only if $X \in Z\left(\Delta_{(v,r)}^s\right)$, for $Z = c_0^F$, c^F and ℓ_∞^F. Also, it is obvious that the norms $\|.\|^{/}$ and $\|.\|^{//}$ are equivalent.

Theorem 3. (*i*) *The spaces* $Sc_0^F\left(\Delta_{(v,r)}^s\right)$, $Sc^F\left(\Delta_{(v,r)}^s\right)$ and $S\ell_\infty^F\left(\Delta_{(v,r)}^s\right)$ *are isometric with the spaces* Sc_0^F, Sc^F *and* $S\ell_\infty^F$.

(*ii*) *The spaces* $Sc_0^F\left(\Delta_{v,r}^s\right)$, $Sc^F\left(\Delta_{v,r}^s\right)$ *and* $S\ell_\infty^F\left(\Delta_{v,r}^s\right)$ *are isometric with the spaces* Sc_0^F, Sc^F *and* $S\ell_\infty^F$.

Proof. (*i*) For $Z = c_0^F$, c^F and ℓ_∞^F, let us define a mapping f: $SZ\left(\Delta_{(v,r)}^s\right) \to SZ$ as follows:

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$$fX = Y = \left(\Delta^{s}_{(v,r)} X_k\right), \text{ for every } X \in SZ\left(\Delta^{s}_{(v,r)}\right) \qquad (3)$$

Then

$$\|fX\|_F = \|Y\|_F = \sup_k \|Y_k\|$$

$= \sup_k \left\|\Delta^{s}_{(v,r)} X_k\right\|$, where $\|.\|_F$ is a norm on SZ, which can be obtained from (2) by taking $s = 0$ and $v_k = 1$ for all $k \in N$.

$= \|X\|^{/}$, using (2).

This completes the proof.

(*ii*) In view of Remark 1, we can also define a mapping similar to (3) on the spaces $Sc_0^F\left(\Delta^{s}_{v,r}\right)$, $Sc^F\left(\Delta^{s}_{v,r}\right)$ and $S\ell_\infty^F\left(\Delta^{s}_{v,r}\right)$. Thus, the proof follows.

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