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On asymptotic statistical equivalence of order α of generalised difference sequences

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Abstract : We introduce and examine the concepts of $\Delta_{uv}^{m}(\lambda)-$asymptotic statistical equivalence of order α, and strong $\Delta_{uv}^{m}(\lambda)-$asymptotic equivalence of order α of sequences**.** Also, we give some relations connected to these concepts.

Keywords: asymptotic equivalence, difference sequences, statistical convergence

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INTRODUCTION

The concept of statistical convergence was introduced by Fast [1] and Schoenberg [2]. Later on it was further investigated from the sequence space point of view and linked with the summability theory by Belen and Mohiuddine [3], Colak [4, 5], Connor [6], Fridy [7], Gadjiev and Orhan [8], Gungor and Et [9], Gungor et al. [10], Isik [11], Kumar and Mursaleen [12], Mohiuddine et al. [13], Mursaleen [14, 15], Rath and Tripathy [16], Šalát [17] and many others. The idea of statistical convergence depends on the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset E of $\mathbb{N}$ is defined by

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k) \;,$$

provided that the limit exists, where χ_E is the characteristic function of the set E. A sequence $x=(x_k)$ is said to be statistically convergent to L if for every $\varepsilon>0$, $\delta(\{k\in\mathbb{N} : |x_k-L|\geq\varepsilon\})=0.$

The concept of asymptotically equivalent sequences was firstly introduced by Pobyvanets [18]. Since the last decades, asymptotically equivalent sequences have been studied by several authors [19-25]. In the present paper, using the generalised difference operator Δ^m and a non-decreasing sequence $\lambda=(\lambda_n)$ of positive real numbers such that $\lambda_{n+1}\leq\lambda_n+1$, $\lambda_1=1$, $\lambda_n\to\infty$

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$(n \to \infty)$, we introduce $\Delta_{uv}^{m}(\lambda)-$asymptotic statistical equivalence of order α and strong $\Delta_{uv}^{m}(\lambda)-$asymptotic equivalence of order α of sequences, and give some relations connected to these concepts.

DEFINITIONS AND PRELIMINARIES

Let w be the set of all sequences of real or complex numbers, and λ_∞, c and c_0 be, respectively, the Banach spaces of bounded, convergent and null sequences $x=(x_k)$ with the usual norm $\|x\| = \sup|x_k|$. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$. The generalised de la Vallée-Poussin mean [26] is defined by

$$t_n(x) = \frac{1}{\lambda_n}\sum_{k\in I_n} x_k,$$

where $I_n = [n-\lambda_n+1, n]$ for $n=1,2,....$ A sequence $x=(x_k)$ is said to be $(V,\lambda)-$summable to a number L if $t_n(x) \to L$ as $n\to\infty$. If $\lambda_n = n$, then $(V,\lambda)-$summability and strong $(V,\lambda)-$summability are reduced to $(C,1)-$summability and $[C,1]-$summability respectively. By Λ, we denote the class of all non-decreasing sequences of positive real numbers tending to ∞ such that $\lambda_{n+1} \le \lambda_n + 1$, $\lambda_1 = 1$.

The notion of difference sequence spaces was introduced by Kizmaz [27] and this concept was generalised by Et and Colak [28]. Later Et and Esi [29] generalised these sequence spaces to the following sequence spaces. Let $u=(u_k)$ be any fixed sequence of non-zero real numbers and let m be a non-negative integer. Then

$$X_u(\Delta^m) = \{x=(x_k) : (\Delta_u^m x_k) \in X\}$$

for $X = \lambda_\infty, c$ or c_0, where $m \in \mathbb{N}$, $\Delta_u^0 x = (u_k x_k)$, $\Delta_u^m x = (\Delta_u^{m-1}x_k - \Delta_u^{m-1}x_{k+1})$, and so $\Delta_u^m x_k = \sum_{i=0}^{m}(-1)^i \binom{m}{i} u_{k+i}x_{k+i}$. The sequence space $X_u(\Delta^m)$ is a Banach space normed by

$$\|x\|_\Delta = \sum_{i=1}^{m}|u_i x_i| + \|\Delta_v^m x_k\|_\infty$$

for $X=\lambda_\infty$, c or c_0. It is noted that the sequence space $X_u(\Delta^m)$ is different from the sequence space $X(\Delta^m)$ and $X_u(\Delta^m) \cap X(\Delta^m) \ne \phi$, where $X(\Delta^m) = \{x=(x_k) : (\Delta^m x_k) \in X\}$. For this, let $X = \lambda_\infty$ and choose $x=(k^{m-1})$ and $u=(k^2)$; then $x \in \lambda_\infty(\Delta^m)$ but $x \notin \lambda_{\infty u}(\Delta^m)$. Conversely if we choose $x=(k^{m+2})$ and $u=(k^{-2})$, then $x\in\lambda_{\infty u}(\Delta^m)$ but $x \notin \lambda_\infty(\Delta^m)$. Let X be any sequence space; if $x \in X_u(\Delta^m)$, then there exists one and only one $z=(z_k)\in X$ such that

$$x_k = u_k^{-1}\sum_{i=1}^{k-m}(-1)^m\binom{k-i-1}{m-1}z_i = u_k^{-1}\sum_{i=1}^{k}(-1)^m\binom{k+m-i-1}{m-1}z_{i-m}, \quad (1)$$
$$z_{1-m} = z_{2-m} = ... = z_0 = 0$$

for sufficiently large k, for example $k > 2m$. We shall use the sequence which is defined in (1) to define the sequence in (2). Recently the difference sequence spaces have been studied [30-37].

MAIN RESULTS

In this section, we give the main results of this paper. In Theorem 1 we give the relationship between $\Delta_{uv}^m(\lambda)-$asymptotic statistical equivalence of order α and $\Delta_{uv}^m(\mu)-$asymptotic statistical

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equivalence of order β of sequences. In Theorem 2 we give the relationship between strong $\Delta_{uv}^{m}(\lambda)-$asymptotic equivalence of order α and strong $\Delta_{uv}^{m}(\mu)-$asymptotic equivalence of order β of sequences. In Theorem 3 we give the relationship between $\Delta_{uv}^{m}(\lambda)-$asymptotic statistical equivalence of order α and strong $\Delta_{uv}^{m}(\mu)-$asymptotic equivalence of order β of sequences.

Two non-negative real-value sequences x and y are said to be Δ^m-asymptotically equivalent provided that

$$\lim_{k}\frac{\Delta^m x_k}{\Delta^m y_k}=L$$

(denoted by $x\overset{\Delta^m}{\sim}y$).

Using the above expression we can make the following definition.

Definition 1. Let $\lambda\in\Lambda$ and $\alpha\in(0,1]$ be any real number. Two non-negative real-value sequences x and y are said to be $\Delta_{uv}^{m}(\lambda)-$asymptotically and statistically equivalent of order α provided that for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{\lambda_n^{\alpha}}\left|\left\{k\in I_n\ :\ \left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|\geq\varepsilon\right\}\right|=0$$

(denoted by $x\overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim}y$), where u and v are two non-negative real-value fixed sequences such that $u_n\neq0$ and $v_n\neq0$ for all $n\in\mathbb{N}$. For $\lambda_n=n$, we shall write $x\overset{S_\alpha^L(\Delta_{uv}^m)}{\sim}y$ instead of $x\overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim}y$ and in the special case $\alpha=1,\ \lambda_n=n,\ u_n=1$ and $v_n=1$ for all $n\in\mathbb{N}$, we shall write $x\overset{S^L(\Delta^m)}{\sim}y$ (which is called Δ^m-asymptotic statistical equivalence) instead of $x\overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim}y$.

It is easy to see that if x and y are Δ^m-asymptotically equivalent, then x and y are Δ^m-asymptotically and statistically equivalent of order α, but the converse does not hold. For this, consider two sequences, $x=(x_k)$ and $y=(y_k)$, defined by

$$\left.\begin{aligned}&\Delta^m x_k=\begin{cases}0, & k\neq n^3\\ 1, & k=n^3\end{cases}\ ,\qquad n=1,2,\ldots\\ &\Delta^m y_k=1,\quad \text{for all}\quad k.\end{aligned}\right\}\qquad(2)$$

It is clear that x and y are Δ^m-asymptotically and statistically equivalent of order α for $\alpha\in(\frac{1}{3},1]$, but they are not Δ^m-asymptotically equivalent.

Definition 2. Let $\lambda\in\Lambda$ and $\alpha\in(0,1]$ be any real number. Two non-negative real-value sequences x and y are said to be strongly $\Delta_{uv}^{m}(\lambda)-$ asymptotically equivalent of order α provided that for every $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{\lambda_n^{\alpha}}\sum_{k\in I_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|=0$$

(denoted by $x\overset{V_\alpha^L[\Delta_{uv}^m(\lambda)]}{\sim}y$). For $\lambda_n=n$, we shall write $x\overset{V_\alpha^L[\Delta_{uv}^m]}{\sim}y$ instead of $x\overset{V_\alpha^L[\Delta_{uv}^m(\lambda)]}{\sim}y$, and in the

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special case $\alpha = 1,\ \lambda_n = n,\ u_n = 1$ and $v_n = 1$ for all $n \in \mathrm{N}$, we shall write $x \overset{V^L[\Delta^m]}{\sim} y$ (which is called strong $\Delta^m -$asymptotic equivalence) instead of $x \overset{V_\alpha^L[\Delta_{uv}^m(\lambda)]}{\sim} y$.

Theorem 1. Let $\lambda, \mu \in \Lambda$ such that $\lambda_n < \mu_n$ for all $n \in \mathrm{N}$, and α and β be fixed real numbers such that $0 < \alpha \le \beta \le 1$. Also, let x and y be two non-negative sequences. Then each of the following assertions holds true:

i) If

$$\liminf_{n\to\infty} \frac{\lambda_n^\alpha}{\mu_n^\beta} > 0 \tag{3}$$

then $x \overset{S_\beta^L(\Delta_{uv}^m(\mu))}{\sim} y$ implies $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$;

ii) If

$$\lim_{n\to\infty} \frac{\mu_n}{\lambda_n^\beta} = 1 \tag{4}$$

then $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ implies $x \overset{S_\beta^L(\Delta_{uv}^m(\mu))}{\sim} y$.

Proof. i) Suppose that $\lambda_n < \mu_n$ for all $n \in \mathrm{N}$ and $x \overset{S_\beta^L(\Delta_{uv}^m(\mu))}{\sim} y$. Then we can write

$$\left\{ k \in J_n \ :\ \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \ge \varepsilon \right\} \supset \left\{ k \in I_n \ :\ \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \ge \varepsilon \right\}$$

and so

$$\frac{1}{\mu_n^\beta} \left| \left\{ k \in J_n \ :\ \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \ge \varepsilon \right\} \right| \ge \frac{\lambda_n^\alpha}{\mu_n^\beta} \frac{1}{\lambda_n^\alpha} \left| \left\{ k \in I_n \ :\ \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \ge \varepsilon \right\} \right|$$

for all $n \in \mathrm{N}$, where $J_n = [n - \mu_n + 1, n]$. Hence we get $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$.

ii) Let $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ and suppose that $I_n \subset J_n$ for all $n \in \mathrm{N}$. Then we can write

$$\begin{aligned}
&\frac{1}{\mu_n^\beta} \left| \left\{ k \in J_n \ :\ \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \ge \varepsilon \right\} \right| \\
&= \frac{1}{\mu_n^\beta} \left| \left\{ n - \mu_n + 1 \le k \le n - \lambda_n \ :\ \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \ge \varepsilon \right\} \right| \\
&\quad + \frac{1}{\mu_n^\beta} \left| \left\{ k \in I_n \ :\ \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \ge \varepsilon \right\} \right| \\
&\le \frac{\mu_n - \lambda_n}{\mu_n^\beta} + \frac{1}{\mu_n^\beta} \left| \left\{ k \in I_n \ :\ \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \ge \varepsilon \right\} \right|
\end{aligned}$$

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$$\le \frac{\mu_n - \lambda_n^{\beta}}{\lambda_n^{\beta}} + \frac{1}{\mu_n^{\beta}} \left| \left\{ k \in I_n \ : \ \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \ge \varepsilon \right\} \right|$$

$$\le \left(\frac{\mu_n}{\lambda_n^{\alpha}} - 1 \right) + \frac{1}{\lambda_n^{\alpha}} \left| \left\{ k \in I_n \ : \ \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \ge \varepsilon \right\} \right|$$

for all $n \in \mathbb{N}$. Now, proceeding to the limit as $n \to \infty$ in the last inequality and using (4), we get $x \overset{S_\beta^L(\Delta_{uv}^m(\mu))}{\sim} y$. The following results are derivable easily from Theorem 1.

Corollary 1. Let $\lambda, \mu \in \Lambda$ be such that $\lambda_n < \mu_n$ for all $n \in \mathbb{N}$, and let x, y be two non-negative sequences. If condition (3) is satisfied, then

i) $x \overset{S_\alpha^L(\Delta_{uv}^m(\mu))}{\sim} y$ implies $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ for each $\alpha \in (0,1]$;

ii) $x \overset{S^L(\Delta_{uv}^m(\mu))}{\sim} y$ implies $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ for each $\alpha \in (0,1]$;

iii) $x \overset{S^L(\Delta_{uv}^m(\mu))}{\sim} y$ implies $x \overset{S^L(\Delta_{uv}^m(\lambda))}{\sim} y$.

Furthermore, if condition (4) is satisfied, then we get following corollary

i) $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ implies $x \overset{S_\alpha^L(\Delta_{uv}^m(\mu))}{\sim} y$ for each $\alpha \in (0,1]$;

ii) $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ implies $x \overset{S^L(\Delta_{uv}^m(\mu))}{\sim} y$ for each $\alpha \in (0,1]$;

iii) $x \overset{S^L(\Delta_{uv}^m(\lambda))}{\sim} y$ implies $x \overset{S^L(\Delta_{uv}^m(\mu))}{\sim} y$.

Theorem 2. Let $\lambda, \mu \in \Lambda$ such that $\lambda_n < \mu_n$ for all $n \in \mathbb{N}$, and α and β be fixed real numbers such that $0 < \alpha \le \beta \le 1$. Also, let x and y be two non-negative sequences. Then each of the following assertions holds true:

i) If condition (3) is satisfied, then $x \overset{V_\beta^L[\Delta_{uv}^m(\mu)]}{\sim} y$ implies $x \overset{V_\alpha^L[\Delta_{uv}^m(\lambda)]}{\sim} y$;

ii) Let $x, y \in \lambda_{\infty u}(\Delta^m)$. If condition (4) is satisfied, then $x \overset{V_\alpha^L[\Delta_{uv}^m(\lambda)]}{\sim} y$ implies $x \overset{V_\beta^L[\Delta_{uv}^m(\mu)]}{\sim} y$,

where $\lambda_{\infty u}(\Delta^m) = \{x = (x_k) \ : \ (\Delta_u^m x_k) \in \lambda_\infty\}$.

Proof. i) Assume that $\lambda, \mu \in \Lambda$ such that $\lambda_n < \mu_n$ for all $n \in \mathbb{N}$ and that condition (3) holds. Since $I_n \subset J_n$, we have

$$\frac{1}{\mu_n^{\beta}} \sum_{k \in J_n} \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \ge \frac{\lambda_n^{\alpha}}{\mu_n^{\beta}} \frac{1}{\lambda_n^{\alpha}} \sum_{k \in I_n} \left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right|$$

Since (3) holds and $x \overset{V_\beta^L[\Delta_{uv}^m(\mu)]}{\sim} y$, we get $x \overset{V_\alpha^L[\Delta_{uv}^m(\lambda)]}{\sim} y$.

ii) Let $x, y \in \lambda_{\infty u}(\Delta^m)$ and suppose that condition (4) holds. Then there exists some $M > 0$ such that $\left| \frac{\Delta_u^m x_k}{\Delta_v^m y_k} - L \right| \le M$ for all k, m. Then we can write

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$$\frac{1}{\mu_n^{\beta}}\sum_{k\in J_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right| = \frac{1}{\mu_n^{\beta}}\sum_{k\in J_n-I_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right| + \frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|$$

$$\leq \left(\frac{\mu_n-\lambda_n}{\mu_n^{\beta}}\right)M + \frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|$$

$$\leq \left(\frac{\mu_n-\lambda_n^{\beta}}{\mu_n^{\beta}}\right)M + \frac{1}{\mu_n^{\beta}}\sum_{k\in I_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|$$

$$\leq \left(\frac{\mu_n}{\lambda_n^{\beta}}-1\right)M + \frac{1}{\lambda_n^{\alpha}}\sum_{k\in I_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|$$

for each $n\in\mathbb{N}$. Therefore $x \overset{V_\beta^L[\Delta_{uv}^m(\mu)]}{\sim} y$. Theorem 2 yields the following corollary.

Corollary 2. Let $\lambda,\mu\in\Lambda$ be such that $\lambda_n<\mu_n$ for all $n\in\mathbb{N}$, and let x,y be two non-negative sequences. If condition (3) is satisfied, then

i) $x \overset{V_\alpha^L[\Delta_{uv}^m(\mu)]}{\sim} y$ implies $x \overset{V_\alpha^L[\Delta_{uv}^m(\lambda)]}{\sim} y$ for each $\alpha\in(0,1]$;

ii) $x \overset{V^L[\Delta_{uv}^m(\mu)]}{\sim} y$ implies $x \overset{V_\alpha^L[\Delta_{uv}^m(\lambda)]}{\sim} y$ for each $\alpha\in(0,1]$;

iii) $x \overset{V^L[\Delta_{uv}^m(\mu)]}{\sim} y$ implies $x \overset{V^L[\Delta_{uv}^m(\lambda)]}{\sim} y$.

Furthermore, if condition (4) is satisfied, then we get the following corollary.

i) $x \overset{V_\alpha^L[\Delta_{uv}^m(\lambda)]}{\sim} y$ implies $x \overset{V_\alpha^L[\Delta_{uv}^m(\mu)]}{\sim} y$ for each $\alpha\in(0,1]$;

ii) $x \overset{V_\alpha^L[\Delta_{uv}^m(\lambda)]}{\sim} y$ implies $x \overset{V^L[\Delta_{uv}^m(\mu)]}{\sim} y$ for each $\alpha\in(0,1]$;

iii) $x \overset{V^L[\Delta_{uv}^m(\lambda)]}{\sim} y$ implies $x \overset{V^L[\Delta_{uv}^m(\mu)]}{\sim} y$.

Theorem 3. Let $\lambda,\mu\in\Lambda$ such that $\lambda_n<\mu_n$ for all $n\in\mathbb{N}$, and α and β be fixed real numbers such that $0<\alpha\leq\beta\leq 1$. Also, let x and y be two non-negative sequences. Then each of the following assertions holds true:

i) If condition (3) is satisfied, then $x \overset{V_\beta^L[\Delta_{uv}^m(\mu)]}{\sim} y$ implies $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$;

ii) Let $x,y\in\lambda_{\infty u}(\Delta^m)$. If condition (4) is satisfied, then $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ implies $x \overset{V_\beta^L[\Delta_{uv}^m(\mu)]}{\sim} y$.

Proof. i) For any two sequences, $x=(x_k)$ and $y=(y_k)$, we can write

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$$\sum_{k\in J_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right| = \sum_{\substack{k\in J_n\\ \left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|\geq\varepsilon}}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right| + \sum_{\substack{k\in J_n\\ \left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|<\varepsilon}}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|$$

$$\geq \sum_{\substack{k\in I_n\\ \left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|\geq\varepsilon}}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|$$

$$\geq \left|\left\{k\in I_n : \left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|\geq\varepsilon\right\}\right|\varepsilon$$

so that

$$\frac{1}{\mu_n^\beta}\sum_{k\in J_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right| \geq \frac{1}{\mu_n^\beta}\left|\left\{k\in I_n : \left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|\geq\varepsilon\right\}\right|\varepsilon$$

$$\geq \frac{\lambda_n^\alpha}{\mu_n^\beta}\frac{1}{\lambda_n^\alpha}\left|\left\{k\in I_n : \left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|\geq\varepsilon\right\}\right|\varepsilon.$$

Hence $x \overset{V_\beta^L[\Delta_{uv}^m(\mu)]}{\sim} y$ implies $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$.

ii) Suppose that $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ and $x, y \in \lambda_{\infty u}(\Delta^m)$. Then there exists some $M > 0$ such that $\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right| \leq M$ for all k, m. Then for every $\varepsilon > 0$ we can write

$$\frac{1}{\mu_n^\beta}\sum_{k\in J_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right| = \frac{1}{\mu_n^\beta}\sum_{k\in J_n - I_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right| + \frac{1}{\mu_n^\beta}\sum_{k\in I_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|$$

$$\leq \left(\frac{\mu_n-\lambda_n}{\mu_n^\beta}\right)M + \frac{1}{\mu_n^\beta}\sum_{k\in I_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|$$

$$\leq \left(\frac{\mu_n-\lambda_n^\beta}{\mu_n^\beta}\right)M + \frac{1}{\mu_n^\beta}\sum_{k\in I_n}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|$$

$$= \left(\frac{\mu_n}{\lambda_n^\beta}-1\right)M + \frac{1}{\mu_n^\beta}\sum_{\substack{k\in I_n\\ \left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|\geq\varepsilon}}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|$$

$$+\frac{1}{\mu_n^\beta}\sum_{\substack{k\in I_n\\ \left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|<\varepsilon}}\left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|$$

$$\leq \left(\frac{\mu_n}{\lambda_n^\beta}-1\right)M + \frac{M}{\lambda_n^\alpha}\left|\left\{k\in I_n : \left|\frac{\Delta_u^m x_k}{\Delta_v^m y_k}-L\right|\geq\varepsilon\right\}\right| + \varepsilon$$

for all $n\in\mathbb{N}$. Using (4), we obtain that $x \overset{V_\beta^L[\Delta_{uv}^m(\mu)]}{\sim} y$ whenever $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$. Corollary 3 below is easily proven by applying Theorem 3.

Corollary 3. Let $\lambda, \mu \in \Lambda$ be such that $\lambda_n < \mu_n$ for all $n\in\mathbb{N}$, and let x, y be two non-negative sequences. If condition (3) is satisfied, then

i) $x \overset{V_\alpha^L[\Delta_{uv}^m(\mu)]}{\sim} y$ implies $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ for each $\alpha \in (0,1]$;

ii) $x \overset{V^L[\Delta_{uv}^m(\mu)]}{\sim} y$ implies $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ for each $\alpha \in (0,1]$;

iii) $x \overset{V^L[\Delta_{uv}^m(\mu)]}{\sim} y$ implies $x \overset{S^L(\Delta_{uv}^m(\lambda))}{\sim} y$.

Furthermore, if condition (4) is satisfied, then we get following corollary.

i) $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ implies $x \overset{V_\alpha^L[\Delta_{uv}^m(\mu)]}{\sim} y$ for each $\alpha \in (0,1]$;

ii) $x \overset{S_\alpha^L(\Delta_{uv}^m(\lambda))}{\sim} y$ implies $x \overset{V^L[\Delta_{uv}^m(\mu)]}{\sim} y$ for each $\alpha \in (0,1]$;

iii) $x \overset{S^L(\Delta_{uv}^m(\lambda))}{\sim} y$ implies $x \overset{V^L[\Delta_{uv}^m(\mu)]}{\sim} y$.

CONCLUSIONS

The results obtained in this study are more general than those reported in the literature. We get several results giving particular values to the numbers m, α, β and the sequences λ, μ, u and v. If we take $\lambda_n = \mu_n$ for all $n \in \mathrm{N}$, then we can write the above theorems without conditions (3) and (4).

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