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Full Paper

Some convolution properties of a subclass of p-valent functions

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Abstract: In this paper we introduce and study a new subclass R_p of p-valent functions that are analytic in the open unit disk $E = \{z: |z| < 1\}$. Some interesting results by using convolution technique for this subclass R_p are obtained. Also, we point out some known consequences of our main results.

Keywords: analytic functions, p-valent functions, p-valently starlike functions, p-valently convex functions, convolution

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INTRODUCTION

Let $A(p)$ denote the class of functions

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \qquad (p \in N = \{1,2,\dots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $E = \{z: |z| < 1\}$. A function $f(z) \in A(p)$ is said to be p-valently starlike of order $\alpha (0 \le \alpha < p)$ in E if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in E).$$

We denote the class of all p-valent starlike functions of order α by $S_p^*(\alpha)$. Further, a function $f(z) \in A(p)$ is said to be p-valently convex of order $\alpha (0 \le \alpha < p)$ in E if and only if

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in E).$$

We denote by $C_p(\alpha)$ the subclass of $A(p)$, consisting of all p-valently convex functions of order α in E. It follows from the definition that if $f(z)$ is p-valently convex function, then $zf'(z)$ is p-valently starlike in E. The classes $S_p^*(\alpha)$ and $C_p(\alpha)$ were first introduced by Owa [1].

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It is easy to see that $S_p^*(0) = S_p^*$ and $C_p(0) = C_p$ are, respectively, the classes of p-valently starlike and p -valently convex functions in E. We also note that $S^*_p = S^*$ and $C_1 = C$ are, respectively, the well-known classes of starlike and convex functions in E.

We say that a function

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \qquad (z \in E)$$

belongs to the class $P(\alpha)$ if $h(z)$ satisfies the following condition

$$Re h(z) > \alpha, \quad 0 \leq \alpha < 1 \quad (z \in E).$$

Let $f(z), g(z) \in A(p)$, where $f(z)$ is given by(1) and $g(z)$ is defined by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \qquad (z \in E).$$

Then the Hadamard product (or convolution) $f * g$ of the functions $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} = (g * f)(z).$$

In recent years many interesting subclasses of analytic multivalent functions associated with the linear operator and their many special cases were investigated by, for example Liu [2] and Sokol et al. [3], and also by Arif et al. [4, 5] and others [e.g. 6-8] using convolution technique.

We now define the following:

Definition 1. A function $f(z)$ given by (1) is said to belong to the class R_p if

$$Re\left(\frac{f^{(p)}(z) + z f^{(p+1)}(z)}{p!}\right) > 0 \quad (z \in E, p \in N = \{1,2,\dots\}), \tag{2}$$

where $f^{(p)}(z)$ is the pth derivative of $f(z)$.

As a special case, the class $R_1 = R$ was studied by Singh and Singh [9] in 1989. Using essentially their technique and that of Lashin [10], we prove the main results for the class R_p, which is the main motivation of this paper.

PRELIMINARY RESULTS

Lemma 1 [11]. Let $\{d_n\}_0^{\infty}$ be a convex null sequence. Then the function

$$q(z) = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n z^n$$

is analytic in E and $Re q(z) > 0 \quad (z \in E)$.

Lemma 2 [12]. If $N(z)$ and $D(z)$ are analytic in E, $N(0) = D(0) = 0$, $D(z)$ is starlike in E and

$$Re\left(\frac{N'(z)}{D'(z)}\right) > 0, \quad \text{then} \quad Re\left(\frac{N(z)}{D(z)}\right) > 0 \quad (z \in E).$$

Lemma 3 [9]. If $h(z)$ is analytic in E, $h(0) = 1$ and $Re h(z) > \frac{1}{2}$ $(z \in E)$, then for any function F analytic in E, the function $h * F$ takes values in the convex hull of the image of E under F.

Lemma 4 [13]. Let $\beta < 1$. If the function $h(z)$ is analytic in E, with $h(0) = 1$, and

$$Re\left(h(z) + z h'(z)\right) > \beta \qquad (z \in E).$$

Then

$$Re h(z) > (2\beta - 1) + 2(1 - \beta)\ln 2 \qquad (z \in E).$$

The result is sharp.

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Lemma 5 [14]. For $\alpha \le 1$ and $\beta \le 1$,

$$P(\alpha) * P(\beta) \subset P(\delta), \qquad \delta = 1 - 2(1-\alpha)(1-\beta).$$

The result is sharp.

MAIN RESULTS

Theorem 1. Let $f(z) \in R_p$; then

$$Re\left(\frac{f^{(p)}(z)}{p!}\right) > -1 + 2log2 \qquad (z \in E).$$

The constant $-1 + 2\log 2$ cannot be replaced by any larger one.

Proof. Let $f(z) \in R_p$; then we have

$$Re\left(\frac{f^{(p)}(z) + zf^{(p+1)}(z)}{p!}\right) > 0 \qquad (z \in E),$$

which can be written as

$$Re\left(1 + \sum_{n=1}^{\infty} \frac{(p+n)!(n+1)}{p!n!} a_{p+n} z^n\right) > 0 \qquad (z \in E), \tag{3}$$

or equivalently,

$$Re\left(1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(p+n)!(n+1)}{p!n!} a_{p+n} z^n\right) > \frac{1}{2} \qquad (z \in E). \tag{4}$$

Consider the function

$$h(z) = 1 + 2\sum_{n=1}^{\infty} \frac{1}{n+1} z^n. \tag{5}$$

Clearly, $h(z)$ is analytic in E, $h(0) = 1$ and

$$\begin{aligned} Reh(z) &= Re\left(1 - \frac{2}{z}\{z + \log(1-z)\}\right) \\ &> -1 + 2log2 \quad [15]. \end{aligned} \tag{6}$$

From (4) and (5) we obtain

$$\frac{f^{(p)}(z)}{p!} = \left(1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(p+n)!(n+1)}{p!n!} a_{p+n} z^n\right) * \left(1 + 2\sum_{n=1}^{\infty} \frac{1}{n+1} z^n\right),$$

from which it follows, in view of (4), (6) and Lemma 3, that

$$Re\left(\frac{f^{(p)}(z)}{p!}\right) > -1 + 2log2 \qquad (z \in E).$$

The constant $-1 + 2\log 2$ cannot be replaced by any larger one, which follows from the fact that the function f_1 defined by $\frac{zf_1^{(p)}(z)}{p!} = -z - 2\log(1-z)$ is in the class R_p.

Corollary 1 [9]. If $f(z) \in R$, then

$$Ref'(z) > -1 + 2log2 = 0.39\ldots \qquad (z \in E).$$

The constant $-1 + 2\log 2$ cannot be replaced by any larger one.

Theorem 2. Let $f(z) \in R_p$, then

$$Re\left(\frac{f^{(p-1)}(z)}{z}\right) > \frac{p!}{2} \qquad (z \in E).$$

Proof. Since the sequence $\{d_n\}_0^{\infty}$ defined by $d_0 = 1, d_n = \frac{2}{(n+1)^2}, n \ge 1$ is a convex null sequence, using Lemma 1 we have

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$$Re\left(1+2\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}z^n\right)>\frac{1}{2}\qquad (z\in E). \tag{7}$$

We can write

$$\frac{f^{(p-1)}(z)}{p!z}=\left(1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(p+n)!(n+1)}{p!n!}a_{p+n}z^n\right)*\left(1+2\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}z^n\right).$$

From (4), (7) and Lemma 3, we have the required result.

Theorem 3. Let $f(z)\in R_p$, then for every $n\geq 1$, the n[th] partial sum of $f(z)$ satisfies $ReS_n^{(p)}(z,f)>0, z\in E$ and hence $S_n(z,f)$ is p-valent in E.

Proof. From (3) and (5) we can write

$$\frac{S_n^{(p)}(z,f)}{p!}=\left(1+\sum_{n=1}^{\infty}\frac{(p+n)!(n+1)}{p!n!}a_{p+n}z^n\right)*\left(1+\sum_{n=1}^{\infty}\frac{1}{(n+1)}z^n\right). \tag{8}$$

Putting $z=re^{i\theta}$, $0\leq r\leq 1$, $0\leq|\theta|\leq\pi$, and using the minimum principle for harmonic functions with the result in the literature[16], we have

$$\begin{aligned}Re\left(1+\sum_{n=1}^{k}\frac{z^n}{n+1}\right)&=Re\left(1+\sum_{n=1}^{k}\frac{r^ne^{in\theta}}{n+1}\right)\\&=Re\left(1+\sum_{n=1}^{k}\frac{r^n}{n+1}(\cos n\theta+i\sin n\theta)\right)\\&=1+\sum_{n=1}^{k}\frac{r^n}{n+1}\cos n\theta\ \ (0\leq\theta\leq\pi)\\&=1+\sum_{n=1}^{k}\frac{\cos n\theta}{n+1}\geq\frac{1}{2}.\end{aligned} \tag{9}$$

Using (3), (8), (9) and Lemma 3, we deduce that $Re\left(S_n^{(p)}(z,f)\right)>0,\ z\in E.$ From the result given [17], we see that $S_n(z,f)$ is p-valent in E for every $n\geq 1$. If $p=1$, then Theorem 2 and Theorem 3 were proved [9].

Theorem 4. If $f(z)\in A(p)$ and

$$Re\left(\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}\right)>-\frac{1}{4}\qquad (z\in E), \tag{10}$$

then $f(z)\in S_p^*(p-1)$.

Proof. Let $f(z)\in A(p)$ given by(1). It follows from the hypothesis of the theorem that

$$Re\left(1+\frac{2}{5}\sum_{n=1}^{\infty}\frac{(p+n)!(n+1)}{p!n!}a_{p+n}z^n\right)>\frac{1}{2}\qquad (z\in E). \tag{11}$$

Also, the sequence $\{d_n\}_0^{\infty}$, where $d_0=1$ and $d_n=\frac{5}{2}\frac{1}{(n+1)^2}, n\geq 1$, is a convex null sequence such as

$$Re\left(1+\frac{5}{2}\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}z^n\right)>\frac{1}{2}\qquad (z\in E). \tag{12}$$

From (11), (12) and Lemma 1, we obtain

$$\begin{aligned}Re\left(\frac{f^{(p-1)}(z)}{p!z}\right)&=Re\left(1+\frac{2}{5}\sum_{n=1}^{\infty}\frac{(p+n)!(n+1)}{p!n!}a_{p+n}z^n\right)*\left(1+\frac{5}{2}\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}z^n\right)\\&>\frac{1}{2}\qquad (z\in E).\end{aligned} \tag{13}$$

Now we define a function $w(z)$ by

$$w(z)=\frac{zf^{(p)}(z)-f^{(p-1)}(z)}{zf^{(p)}(z)+f^{(p-1)}(z)},$$

which can be written as

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$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} = \frac{1+w(z)}{1-w(z)} \qquad (z \in E). \tag{14}$$

Clearly, $w(0) = 0$. Since $f(z)$ is p-valent in E, we have $w(z) \neq 1$ in E. From (14) we obtain

$$\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!} = \frac{f^{(p-1)}(z)}{p!z}\left(\left(\frac{1+w(z)}{1-w(z)}\right)^2 + \frac{2zw'(z)}{(1-w(z))^2}\right). \tag{15}$$

We claim that $|w(z)| < 1$ in E. If this is not true, then there exists a point $z_0 \in E$ such that

$$max_{|z|\le|z_0|}|w(z)| = |w(z_0)| = 1.$$

Then from the result given [18], we have

$$zw'(z_0) = kw(z_0), \text{where } k \ge 1, w(z_0) = e^{i\theta}, 0 < \theta < 2\pi.$$

Putting $z = z_0$ in (15), we obtain

$$Re\left(\frac{f^{(p)}(z_0)+z_0f^{(p+1)}(z_0)}{p!}\right) = Re\left(\frac{f^{(p-1)}(z_0)}{p!z_0}\left(\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^2 + \frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right)\right)$$

$$\le -\frac{\cos\theta+1+k}{\cos\theta-1} Re\left(\frac{f^{(p-1)}(z_0)}{(z_0)}\right)$$

$$\le -\frac{k}{2} Re\left(\frac{f^{(p-1)}(z_0)}{(z_0)}\right)$$

$$\le -\frac{1}{4}. \tag{16}$$

Since $k \ge 1$ and from (13), $Re\left(\frac{f^{(p-1)}(z)}{p!z}\right) > \frac{1}{2}$, $z \in E$. Inequality (16) contradicts inequality (10); thus, $|w(z)| < 1$ in E. Equation (14) then implies that $f(z) \in S_p^*(p-1)$ [19].

Using the Alexander type relation, we obtain the following corollary.

Corollary 2. If $g(z) \in A(p)$ and

$$Re\left(\frac{g^{(p)}(z)+3zg^{(p+1)}(z)+z^2g^{(p+2)}(z)}{p!}\right) > -\frac{1}{4} \qquad (z \in E),$$

then $g(z) \in C_p(p-1)$.

Corollary 3 [9]. If $f(z) \in A$ and let

$$Re(f'(z) + zf''(z)) > -\frac{1}{4} \qquad (z \in E),$$

then $f(z) \in S^*$.

Our next result shows that the class R_p is closed with respect to Hadamard product.

Theorem 5. If $f(z)$ and $g(z)$ belong to the class R_p and

$$h^{(p-1)}(z) = f^{(p-1)}(z) * g^{(p-1)}(z),$$

then $h(z)$ also belongs to the class R_p.

Proof. Since

$$h^{(p-1)}(z) = f^{(p-1)}(z) * g^{(p-1)}(z),$$

it follows that

$$zh^{(p)}(z) = zf^{(p)}(z) * g^{(p-1)}(z).$$

A simple computation gives

$$Re\left(\frac{h^{(p)}(z)+zh^{(p+1)}(z)}{p!}\right) = Re\left(\left(\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}\right) * \left(\frac{g^{(p-1)}(z)}{p!z}\right)\right). \tag{17}$$

From (17), using (1), (13) and Lemma 3, we have the desired result.

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Corollary 4 [9]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to R, then their Hadamard product

$$h(z) = (f * g)(z)$$

also belongs to R.

From the proof of Theorem 5, it is clear that the following more general result holds.

Theorem 6. If $f(z) \in R_p$, $g(z) \in A(p)$ and $Re\left(\frac{g^{(p-1)}(z)}{p!z}\right) > \frac{1}{2}$, $z \in E$, then

$$f^{(p-1)}(z) * g^{(p-1)}(z)$$

also belongs to the class R_p.

Theorem 7. Let $f(z) \in R_p$. Then $Re(f^{(p)}(z)) > 0$, $z \in E$ and hence $f(z)$ is p-valent in E.

Proof. Using the result given [17] and applying Lemma 2 with $N(z) = \frac{zf^{(p)}(z)}{p!}$ and $D(z) = z$ proves Theorem 7. If $p = 1$, then Theorem 7 was proved earlier [20].

Ruscheweyh and Small [21] have proved that the class C is closed with respect to Hadamard product. In what follows we prove that the Hadamard product of functions of the class R_p belongs to the class $C_p(\alpha)$.

Theorem 8. If $f(z)$ and $g(z)$ belong to R_p and

$$h^{(p-1)}(z) = f^{(p-1)}(z) * g^{(p-1)}(z),$$

then $h(z) \in C_p(p-1)$.

Proof. In view of Corollary 3, it is sufficient to show that

$$Re\left(\frac{h^{(p)}(z)+3zh^{(p+1)}(z)+z^2h^{(p+2)}(z)}{p!}\right) > -\frac{1}{4} \qquad (z \in E).$$

Equivalently, this can be written as

$$Re\left(1 + \sum_{n=1}^{\infty} (n+1)\left(\frac{(n+p)!}{n!p!}\right)^2 a_{n+p} b_{n+p} z^n\right) > -\frac{1}{4} \qquad (z \in E). \tag{18}$$

Since $f(z), g(z) \in R_p$, we have

$$Re\left(1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(n+p)!(n+1)}{p!n!} a_{n+p} z^n\right) > \frac{1}{2} \qquad (z \in E),$$

and

$$Re\left(1 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(n+p)!(n+1)}{p!n!} b_{n+p} z^n\right) > \frac{1}{2} \qquad (z \in E).$$

Using Lemma 3, we obtain

$$Re\left(1 + \frac{1}{4}\sum_{n=1}^{\infty} (n+1)^2\left(\frac{(n+p)!}{n!p!}\right)^2 a_{n+p} b_{n+p} z^n\right) > \frac{1}{2} \qquad (z \in E). \tag{19}$$

Consider the function

$$h(z) = \left(1 + 4\sum_{n=1}^{\infty} \frac{1}{n+1} z^n\right). \tag{20}$$

Clearly, $h(z)$ is analytic in E, $h(0) = 1$ and

$$\begin{aligned} Re\left(1 + 4\sum_{n=1}^{\infty} \frac{1}{n+1} z^n\right) &= Re\left(-3 - \frac{4}{z}\log(1-z)\right) \\ &> -3 + 4log2 \ [15] \\ &> -\frac{1}{4} \qquad (z \in E). \end{aligned} \tag{21}$$

From (19) and (20), we can write

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$$\left(1+\sum_{n=1}^{\infty}(n+1)\left(\frac{(n+p)!}{n!p!}\right)^2 a_{n+p}b_{n+p}z^n\right)$$

$$=\left(1+\frac{1}{4}\sum_{n=1}^{\infty}(n+1)^2\left(\frac{(n+p)!}{n!p!}\right)^2 a_{n+p}b_{n+p}z^n\right)*\left(1+4\sum_{n=1}^{\infty}\frac{1}{n+1}z^n\right). \qquad (22)$$

Using (19), (21), (22) and Lemma 3, we see that (18) holds for all $z\in E$. This completes the proof.

Corollary 5 [9]. If $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ belong to R, then

$$h(z)=(f*g)(z)\in C.$$

We now define the integral operator $F(z)$ as follows. Let $f(z)\in A(p)$; then

$$\frac{F^{(p-1)}(z)}{p!}=\frac{c+1}{z^c}\int_0^z t^{c-1}\left(\frac{f^{(p-1)}(t)}{p!}\right)dt, \qquad (c>-p,\ p\in N=\{1,2,\dots\}).$$

$$=z+\sum_{n=1}^{\infty}\frac{(c+1)(n+p)!}{(n+c+1)(n+1)!p!}a_{n+p}z^{n+1}. \qquad (23)$$

For p=1, the operator defined in (23) is a generalised form of operator by Bernardi [22]. A comprehensive study of operators with applications can be found in the literature [23-30].

Theorem 9. Let $f(z)\in R_p$ and

$$\frac{F^{(p-1)}(z)}{p!}=\frac{c+1}{z^c}\int_0^z t^{c-1}\left(\frac{f^{(p-1)}(t)}{p!}\right)dt \qquad (z\in E). \qquad (24)$$

Then $F(z)\in R_p$.

Proof. Let

$$\frac{F^{(p)}(z)+zF^{(p+1)}(z)}{p!}=h(z).$$

Then $h(z)$ is analytic in E and $h(0)=1$.
From (24), we have

$$\left(\frac{z^cF^{(p-1)}(z)}{p!}\right)'=(c+1)z^{c-1}\frac{f^{(p-1)}(z)}{p!}.$$

A simple computation gives us:

$$Re\left(h(z)+\frac{zh'(z)}{c+1}\right)=Re\left(\frac{f^{(p)}(z)+zf^{(p+1)}(z)}{p!}\right).$$

From the hypothesis of Theorem 9 with the result given [31], we have

$$Re\left(\frac{F^{(p)}(z)+zF^{(p+1)}(z)}{p!}\right)>0 \qquad (z\in E).$$

This completes the proof. It is easy to see that if $0\le\lambda\le 1$, and $f(z)$ and $g(z)$ are in R_p, then $G(z)=\lambda g(z)+(1-\lambda)f(z)$ is also in R_p. This shows that the class R_p is a convex set.

Theorem 10. Let $f(z), g(z)\in A(p)$ and $\alpha,\beta<1$. If

$$\frac{f^{(p)}(z)}{p!}\in P(\alpha), \quad \frac{g^{(p)}(z)}{p!}\in P(\beta),$$

and

$$\phi^{(p-1)}(z)=f^{(p-1)}(z)*g^{(p-1)}(z),$$

then $\phi(z)\in S_p^*(p-1)$, provided that

$$(1-\alpha)(1-\beta)<\frac{3}{8(\ln 2-1)^2+4}. \qquad (25)$$

Proof. Using the given hypothesis on $f(z)$ and $g(z)$ and Lemma 5, we have

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$$Re\left(\frac{f^{(p)}(z)}{p!} * \frac{g^{(p)}(z)}{p!}\right) = Re\left(\frac{\phi^{(p)}(z)+z\phi^{(p+1)}(z)}{p!}\right)$$
$$> 1-2(1-\alpha)(1-\beta). \quad (26)$$

By using Lemma 4, from (26) we have

$$Re\left(\frac{\phi^{(p)}(z)}{p!}\right) > 1+4(1-\alpha)(1-\beta)(\ln 2-1) \qquad (z \in E),$$

or equivalently, this can be written as

$$Re\left(\frac{\phi^{(p-1)}(z)}{p!z} + z\left(\frac{\phi^{(p-1)}(z)}{p!z}\right)'\right) > 1+4(1-\alpha)(1-\beta)(\ln 2-1) \quad (z \in E). \quad (27)$$

Using Lemma 4, (27) becomes

$$Re\left(\frac{\phi^{(p-1)}(z)}{p!z}\right) > 1-8(1-\alpha)(1-\beta)(\ln 2-1)^2.$$

Suppose

$$h(z) = \frac{z\phi^{(p)}(z)}{\phi^{(p-1)}(z)} \text{ and } q(z) = \frac{\phi^{(p-1)}(z)}{z}.$$

Then $h(z)$ is analytic in E, $h(0)=1$ and

$$Req(z) > 1-8(1-\alpha)(1-\beta)(\ln 2-1)^2. \quad (28)$$

A simple computation gives us:

$$\phi^{(p)}(z) + z\phi^{(p+1)}(z) = q(z)(h^2(z) + zh'(z))$$
$$= \Psi(h(z), zh'(z), z). \quad (29)$$

By taking $u = h(z)$ and $v = zh'(z)$, $\Psi(u, v; z) = q(z)(u^2 + v)$. From (25) and (29) we have

$$Re\left(\Psi(h(z), zh'(z), z)\right) > 1-2(1-\alpha)(1-\beta) \qquad (z \in E).$$

Now for real $x, y \le -\frac{1}{2}(1+x^2)$, we have

$$Re\left(\Psi(ix, y, z)\right) = (-x^2+y)Req(z)$$
$$\le -\frac{1}{2}(1+3x^2)Req(z)$$
$$\le -\frac{1}{2}Req(z) \qquad (z \in E). \quad (30)$$

From (28) and (30) we obtain

$$Re\,(\Psi(ix, y, z)) \le 1-2(1-\alpha)(1-\beta), \quad \text{for all } z \in E.$$

By using the results given [19, 31], we have $\phi(z) \in S_p^*(p-1) \quad (z \in E)$.

Corollary 6. Let $f(z), g(z) \in A(p)$ and $\alpha, \beta < 1$. If

$$\frac{f^{(p)}(z)}{p!} \in P(\alpha), \quad \frac{g^{(p)}(z)}{p!} \in P(\beta),$$

and

$$\psi^{(p-1)}(z) = \int_0^z \frac{(f^{(p-1)} * g^{(p-1)})(t)}{t} dt,$$

then $\psi(z) \in C_p(p-1)$, provided that

$$(1-\alpha)(1-\beta) < \frac{3}{8(\ln 2-1)^2+4}.$$

The proof is simple by taking $z\psi^{(p)}(z) = \phi^{(p-1)}(z)$.

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Theorem 11. Let $f(z)$, $g(z)$, $h(z)\in A(p)$, α, β, $\gamma < 1$. If

$$\frac{f^{(p)}(z)}{p!} \in P(\alpha),\ \frac{g^{(p)}(z)}{p!} \in P(\beta),\ \frac{h^{(p)}(z)}{p!} \in P(\gamma),$$

and

$$\emptyset^{(p-1)}(z) = (f^{(p-1)} * g^{(p-1)} * h^{(p-1)})(z),$$

then $\emptyset(z)\in S_p^*(p-1)$, provided that

$$(1-\alpha)(1-\beta)(1-\gamma) < \frac{3}{\{8(\ln 2-1)^2+4\}\{(-4)(\ln 2-1)\}}.$$

Proof. By the hypotheses on *f, g* and *h* and Lemma 5, we obtain

$$Re\left(\frac{K^{(p)}(z)}{p!} * \frac{h^{(p)}(z)}{p!}\right) = Re\left(\frac{\emptyset^{(p)}(z) + z\emptyset^{(p+1)}(z)}{p!}\right)$$
$$> 1 - 2(1-\alpha_1)(1-\beta), \qquad (31)$$

where $K^{(p-1)}(z) = \left(f^{(p-1)} * g^{(p-1)}\right)(z)$ and $Re\frac{K^{(p)}(z)}{p!} > \alpha_1, \alpha_1 = 1 + 4(1-\alpha)(1-\beta)(\ln 2 - 1)$.

From (31), together with Lemma 4, we have

$$Re\left(\frac{\emptyset^{(p)}(z)}{p!}\right) > 1 - 16(1-\alpha)(1-\beta)(1-\gamma)(\ln 2-1)^2 \qquad (z \in E),$$

Using the same technique similar to that of Thereom 10, we obtain the required result.

Corollary 7. Let $f(z)$, $g(z)$, $h(z)\in A(p)$, α, β, $\gamma < 1$. If

$$\frac{f^{(p)}(z)}{p!} \in P(\alpha),\ \frac{g^{(p)}(z)}{p!} \in P(\beta),\ \frac{h^{(p)}(z)}{p!} \in P(\gamma),$$

and

$$\varphi^{(p-1)}(z) = (f^{(p-1)} * g^{(p-1)} * h^{(p-1)})(z),$$

Then $\varphi(z)\in C_p(p-1)$, provided that

$$(1-\alpha)(1-\beta)(1-\gamma) < \frac{3}{16(\ln 2-1)^2+8}.$$

For proving $\varphi(z)\in C_p(p-1)$, it is sufficient to show that

$$\zeta^{(p-1)}(z) = z\varphi^{(p)}(z) \in S_p^*(p-1).$$

By the hypotheses on $f(z)$, $g(z)$ and $h(z)$ and Lemma 5, we obtain

$$Re\left(\frac{(f^{(p)} * g^{(p)} * h^{(p)})(z)}{p!}\right) = Re\left(\frac{\zeta^{(p)}(z) + z\zeta^{(p+1)}(z)}{p!}\right)$$
$$> 1 - 4(1-\alpha)(1-\beta)(1-\gamma),$$

and the proof is completed similarly to that of Theorem 10. If $p = 1$, then Theorem 10 and Corollary 7 were given [10].

Theorem 12. Let $f_1(z), f_2(z), \ldots, f_n(z)\in A(p)$, $\alpha_1, \alpha_1, \ldots, \alpha_n < 1$. If

$$\frac{f_1^{(p)}(z)}{p!} \in P(\alpha_1),\ \frac{f_2^{(p)}(z)}{p!} \in P(\alpha_2), \ldots, \frac{f_n^{(p)}(z)}{p!} \in P(\alpha_n),$$

and

$$\tau^{(p-1)}(z) = (f_1^{(p-1)} * f_2^{(p-1)} * \ldots * f_n^{(p-1)})(z), \qquad (32)$$

then $\tau(z)\in S^*_p(p-1)$, provided that

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$$(1-\alpha_1)(1-\alpha_2)...(1-\alpha_n) < \frac{3}{\{8(\ln 2-1)^2+4\}\{(-4)(\ln 2-1)\}^{n-2}}, \quad n \geq 2. \tag{33}$$

Proof. For proving the above Theorem, we use the principle of mathematical induction. For $n = 2$, we have proved Theorem 10; thus, (32) holds for $n = 2$. Suppose that (32) holds true for $n = k$; that is,

$$\tau^{(p-1)}(z) = (f_1^{(p-1)} * f_2^{(p-1)} * ... * f_k^{(p-1)})(z),$$

then $\tau(z) \in S^*p(p-1)$, provided that inequality (33) is satisfied.

We have to prove that (32) holds true for $n = k+1$. For this, consider

$$\tau^{(p-1)}(z) = (f_1^{(p-1)} * f_2^{(p-1)} * ... * f_{k+1}^{(p-1)})(z).$$

Now using the given hypothesis on $fj(z)$, j = 1,2, ... , k and Lemma 5, we have

$$Re\left(\frac{M^{(p)}(z)}{p!} * \frac{f_{k+1}{}^{(p)}(z)}{p!}\right) = Re\left(\frac{\tau^{(p)}(z) + z\tau^{(p+1)}(z)}{p!}\right)$$
$$> 1 - 2(1-\alpha^*)(1-\alpha_{k+1}), \tag{34}$$

where $M^{(p-1)}(z) = (f1^{(p-1)} * f2^{(p-1)} * ... * fk^{(p-1)})$ and

$$Re\frac{M^{(p)}(z)}{p!} > \alpha^*, \ \alpha^* = 1 + 4(1-\alpha_1)(1-\alpha_2)...(1-\alpha_k)(ln2-1)^{k-1}(-4)^{k-2}.$$

By using Lemma 4, from (34) we have

$$Re\frac{\tau^{(p)}(z)}{p!} > 1 + 4(1-\alpha_1)(1-\alpha_2)...(1-\alpha_k)(ln2-1)^{k}(-4)^{k-1}, \ z \in E, \ k \geq 2. \tag{35}$$

Applying Lemma 4, (35) can be written as

$$Re\frac{\tau^{(p-1)}(z)}{zp!} > 1 - 8(1-\alpha^*)(1-\alpha_{k+1}).$$

Now with the same procedure used in Theroem 10, we have $\tau(z) \in S^*p(p-1)$, provided that

$$(1-\alpha_1)(1-\alpha_2)...(1-\alpha_k)(1-\alpha_{k+1}) < \frac{3}{\{8(\ln 2-1)^2+4\}\{(-4)(\ln 2-1)\}^{k-1}}.$$

Therefore, the result is true for $n = k + 1$ and hence by using mathematical induction, (32) holds true for all $n \geq 2$. This completes the proof.

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